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On the expansion of Fibonacci and Lucas polynomials, revisited

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Abstract: As established by Prodinger in "On the Expansion of Fibonacci and Lucas Polynomials", we give q-analogue of identities established by Belbachir and Bencherif in "On some properties of bivariate Fibonacci and Lucas polynomials". This is doing according to the recent Cigler's definition for the q-analogue of Fibonacci polynomials, given in "Some beautiful q-analogues of Fibonacci and Lucas polynomials", and by the authors for the q-analogues of Lucas polynomials, given in "An Alternative approach to Cigler's q-Lucas polynomials".

Keywords: Fibonacci Polynomials; Lucas Polynomials; q-analogue.

Résumé : Comme établi par Prodinger dans "On the Expansion of Fibonacci and Lucas Polynomials", nous donnons le q-analogue des identités établies par Belbachir et Bencherif dans "On some properties of bivariate Fibonacci and Lucas polynomials". Ces identités sont basées sur l'approche de Cigler pour le q-analogue des polynômes de Fibonacci, donnée dans "Some beautiful q-analogues of Fibonacci and Lucas polynomials", et par les auteurs pour les q-analogues de polynômes de Lucas, donnée dans "An Alternative approach to Cigler's q-Lucas polynomials".

Mots clés : Polynômes de Fibonacci; Polynômes Lucas; q-analogue.

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1 Introduction

The bivariate polynomials of Fibonacci and Lucas, denoted respectively by (U_n) and (V_n), are defined by

$$\begin{cases} U_0 = 0,\ U_1 = 1, \\ U_n = tU_{n-1} + zU_{n-2} \quad (n \geq 2), \end{cases} \quad \text{and} \quad \begin{cases} V_0 = 2,\ V_1 = t, \\ V_n = tV_{n-1} + zV_{n-2} \ \ (n \geq 2). \end{cases}$$

It is established, see for instance [1], that

$$U_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} t^{n-2k} z^k, \qquad V_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} t^{n-2k} z^k \ \ (n \geq 1).$$

In [2], the first author and Bencherif proved that, for $n - 2\lfloor n/2 \rfloor \leq k \leq n - \lfloor n/2 \rfloor$, the families $\left(x^k U_{n+1-k}\right)_k$ and $\left(x^k V_{n-k}\right)_k$ constitute two basis of the $\mathbb{Q}$-vector space spanned by the free family $\left(x^{n-2k} y^k\right)_k$, and they found that the coordinates of the bivariate polynomials of Fibonacci and Lucas, over appropriate basis, satisfies remarkable recurrence relations. They established the following formulae

$$V_{2n} = 2U_{2n+1} - xU_{2n}, \tag{1}$$

$$2U_{2n+1} = \sum_{k=0}^{n} a_{n,k} t^k V_{2n-k}, \ \text{with} \ \ a_{n,k} = 2\sum_{j=0}^{n} (-1)^{j+k} \binom{j}{k} - (-1)^{n+k} \binom{n}{k}, \tag{2}$$

$$V_{2n} = \sum_{k=1}^{n} b_{n,k} t^k V_{2n-k}, \ \text{with} \ \ b_{n,k} = (-1)^{k+1} \binom{n}{k}, \tag{3}$$

$$V_{2n-1} = \sum_{k=1}^{n} c_{n,k} t^k U_{2n-k}, \ \text{with} \ \ c_{n,k} = 2(-1)^{k+1} \binom{n}{k} - [k=1], \tag{4}$$

$$2V_{2n-1} = \sum_{k=1}^{n} d_{n,k} t^k V_{2n-1-k}, \ \text{with} \ \ d_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k}, \tag{5}$$

$$2U_{2n} = \sum_{k=1}^{n} e_{n,k} t^k V_{2n-k}, \ \text{with} \tag{6}$$

$$e_{n,k} = (-1)^{k+1} \frac{2n-k}{n} \binom{n}{k} + \sum_{j=0}^{n-1} (-1)^{j+k-1} \binom{j}{k-1} - \frac{1}{2} (-1)^{n+k} \binom{n-1}{k-1}.$$

A similar approach was done by the authors, see [3], for Chebyshev polynomials.

As q-analogue of Fibonacci and Lucas polynomials, J. Cigler [6], considers the following expressions

$$\mathbf{F}_{n+1}(x, y, m) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2} + m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q x^{2n-k} y^k, \tag{7}$$

$$\mathbf{Luc}_n(x, y, m) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{[n]_q}{[n-k]_q} x^{2n-k} y^k, \tag{8}$$

with the q-notations

$$[n]_q = 1 + q + \cdots + q^{n-1},\ [n]_q! = [1]_q[2]_q \cdots [n]_q,\ \begin{bmatrix} n-k \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Without loss the generality, we can suppose that $t = 1$. We refer here to the modified polynomials given by H. Prodinger in the introduction of [8].

Using the q-analogues of Fibonacci and Lucas polynomials suggested by J. Cigler, H. Prodinger, see [8], give q-analogues for relations (3) and (5), and the authors, see [4], give q-analogues for relations (1), (2), (4), (6).

The q-identities associated to relations (2) and (6), given in [4], do not give for $q = 1$ the initial relations. This is the motivation which conclude to this paper: we propose an alternative q-analogue for all the former relations (1), (2), (3), (4), (5) and (6) based on Cigler's definition, see [6], for the Fibonacci polynomials, and the definitions givn by the authors, see [5], for the Lucas polynomials.

In [5], we have defined the q-Lucas polynomials of the first kind $\mathbf{L}(z)$ and the q-Lucas polynomials of the second kind $\mathbb{L}(z)$ respectively by

$$\mathbf{L}_n(z,m) \ :=\ \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + \frac{[k]_q}{[n-k]_q}\right) z^k, \tag{9}$$

$$\mathbb{L}_n(z,m) \ :=\ \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{k+1}{2}+m\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \left(1 + q^{n-2k}\frac{[k]_q}{[n-k]_q}\right) z^k, \tag{10}$$

and we have showed that the polynomials $\mathbf{L}_n(z)$ and $\mathbb{L}_n(z)$ satisfy the recursions

$$\mathbf{L}_{n+1}(z,m) = \mathbf{L}_n(z,m) + q^{n-1} z \mathbf{L}_{n-1}\left(q^{m-1}z, m\right), \tag{11}$$

$$\mathbb{L}_{n+1}(z,m) = \mathbb{L}_n(qz,m) + qz\mathbb{L}_{n-1}\left(q^{m+1}z, m\right). \tag{12}$$

These two recursions are satisfied by the q-analogue of Fibonacci polynomials $\mathbf{F}_n(z,m)$, see [6].

2 Main results

In [5], the authors expressed the q-Lucas polynomials of the both kinds in terms of q-Fibonacci polynomials by the identities

$$\begin{aligned} \mathbf{L}_n(z,m) &= 2\mathbf{F}_{n+1}\left(\frac{z}{q}, m\right) - \mathbf{F}_n(z,m), \\ \mathbb{L}_n(z,m) &= 2\mathbf{F}_{n+1}(z,m) - \mathbf{F}_n(z,m), \end{aligned}$$

which are considered as a q-analogue of the equality (1).

The following result gives two q-analogues of relation (2), the first one is related to the q-Lucas polynomials of the first kind and the second one is related to the q-Lucas polynomials of the second kind.

Theorem 1 *For every integer $n \geq 0$, one has*

$$\begin{aligned} 2\mathbf{F}_{2n+1}\left(\frac{z}{q}, m\right) &= \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n+k} \mathbf{L}_{2n-k}(z, m) + \\ & 2\sum_{j=0}^{n-1}\sum_{k=0}^{j} q^{\binom{k}{2}-2nj} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{k+j} \mathbf{L}_{2n-k}\left(q^{2j}z, m\right), \end{aligned} \tag{13}$$

$$\begin{aligned} 2\mathbf{F}_{2n+1}(z, m) &= 2\sum_{j=0}^{n-1}\sum_{k=0}^{j} q^{\binom{j-k}{2}-\binom{j}{2}+2nj} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{k+j} \mathbb{L}_{2n-k}\left(q^{k-2j}z, m\right) + \\ & \sum_{k=0}^{n} q^{n(n-k)+\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k} \mathbb{L}_{2n-k}\left(q^{k-n}z, m\right). \end{aligned} \tag{14}$$

The q-analogue of relation (3), found by Prodinger in [8], is given by the following Theorem which gives also a second q-analogue identity.

Theorem 2 *For $n \geq 1$, we have*

$$\mathbf{F}_{2n}(z, m) = \sum_{j=1}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j+1} \mathbf{F}_{2n-j}(z, m), \tag{15}$$

$$\mathbf{F}_{2n}(z, m) = \sum_{j=1}^{n} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j+1} \mathbf{F}_{2n-j}\left(q^{j}z, m\right). \tag{16}$$

The relation (4), admits two q-analogues related to the q-Lucas polynomials of the first kind $\mathbf{L}(z)$, and two q-analogues related to the q-Lucas polynomials of the second kind $\mathbb{L}(z)$, given respectively by the following Theorem.

Theorem 3 *For $n \geq 1$, the q-Lucas polynomials of the first kind is developed by*

$$\mathbf{L}_{2n-1}(z, m) = 2\sum_{j=1}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j+1} \mathbf{F}_{2n-j}\left(\frac{z}{q}, m\right) - \mathbf{F}_{2n-1}(z, m), \tag{17}$$

$$\mathbf{L}_{2n-1}(z, m) = 2\sum_{j=1}^{n} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j+1} \mathbf{F}_{2n-j}\left(q^{j-1}z, m\right) - \mathbf{F}_{2n-1}(z, m). \tag{18}$$

and the q-Lucas polynomials of the second kind is developed by

$$\mathbb{L}_{2n-1}(z, m) = 2\sum_{j=1}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j+1} \mathbf{F}_{2n-j}(z, m) - \mathbf{F}_{2n-1}(z, m), \tag{19}$$

$$\mathbb{L}_{2n-1}(z, m) = 2\sum_{j=1}^{n} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j+1} \mathbf{F}_{2n-j}\left(q^{j}z, m\right) - \mathbf{F}_{2n-1}(z, m). \tag{20}$$

Using the q-Lucas polynomials of the both kinds we find two q-analogues of relation (5).

Theorem 4 *For $n \geq 1$, we have*

$$2\mathbf{L}_{2n-1}(z,m) = \sum_{k=1}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k+1} q^{k+1} \left(1 + \frac{[n-k]_q}{[n]_q}\right) \mathbf{L}_{2n-2-k}(z,m), \qquad (21)$$

$$2\mathbb{L}_{2n-1}(z,m) = \sum_{k=1}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k+1} q^{1-n} \left(1 + q^k \frac{[n-k]_q}{[k]_q}\right) \mathbb{L}_{2n-1-k}\left(q^k z, m\right) \qquad (22)$$

In the following theorem, we give two q-analogues of (6).

Theorem 5 *For every integer $n \geq 0$, one has*

$$\begin{aligned} & 2\mathbf{F}_{2n}\left(\frac{z}{q}, m\right) \\ = & \frac{1}{2}\sum_{k=1}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k+1} q^{k+1} \left(1 + \frac{[n-k]_q}{[n]_q}\right) \mathbf{L}_{2n-2-k}(z,m) + \\ & \frac{1}{2}\sum_{k=0}^{n-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^{n-1+k} \mathbf{L}_{2n-2-k}(qz,m) + \\ & \sum_{j=0}^{n-2}\sum_{k=0}^{j} q^{\binom{k}{2}-2nj-2j} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{k+j} \mathbf{L}_{2n-2-k}\left(q^{2j+1}z, m\right). \end{aligned} \qquad (23)$$

and

$$\begin{aligned} & 2\mathbf{F}_{2n}(z) \\ = & \frac{1}{2}\sum_{k=1}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k+1} q^{1-n} \left(1 + q^k \frac{[n-k]_q}{[k]_q}\right) \mathbb{L}_{2n-1-k}\left(q^k z, m\right) + \\ & \frac{1}{2}\sum_{k=0}^{n} q^{(n-1)(n-k)+\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^{n-1+k} \mathbb{L}_{2n-2-k}\left(q^{k+1-n}z, m\right) + \\ & \sum_{j=0}^{n-1}\sum_{k=0}^{j} q^{\binom{j-k}{2}-\binom{j}{2}+2nj-2j} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{k+j} \mathbb{L}_{2n-2-k}\left(q^{k-2j}z, m\right). \end{aligned} \qquad (24)$$

Remark 1 *Notice that the coefficients appeared in the sums given in the different results do not depends on m.*

3 Proof of the results

We need the following Lemmas.

Lemma 6 *For $U_n(z)$ satisfying (11) and (12) respectively, we have for $k \geq 1$*

$$\begin{aligned}\sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+k-j}(z,m) &= q^{m\binom{k}{2}+\binom{n}{2}-\binom{n-k}{2}} z^k U_{n-k}\left(q^{mk-k}z,m\right), \\ \sum_{j=0}^{k} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+k-j}\left(q^{j-k}z,m\right) &= q^{m\binom{k}{2}} z^k U_{n-k}\left(q^{mk}z,m\right).\end{aligned}$$

Proof. We use induction over k, the case $k=1$ is given by the recursions (11) and (12). We suppose the relation true for k,

$$\begin{aligned}&q^{m\binom{k+1}{2}+\binom{n}{2}-\binom{n-1-k}{2}} z^{k+1} U_{n-k-1}\left(q^{mk+m-1-k}z,m\right) \\ &= q^{m\binom{k}{2}+\binom{n}{2}-\binom{n-k}{2}} q^k z^k \left(U_{n+1-k}\left(q^{mk-k}z,m\right) - U_{n-k}\left(q^{mk-k}z,m\right)\right) \\ &= q^{m\binom{k}{2}+\binom{n-1}{2}-\binom{n-1-k}{2}} z^k U_{n+1-k}\left(q^{mk-k}z,m\right) \\ &\quad -q^{m\binom{k}{2}+\binom{n}{2}-\binom{n-k}{2}} q^k z^k U_{n-k}\left(q^{mk-k}z,m\right) \\ &= \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+1-k-j}(z,m) - q^k \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+k-j}(z,m) \\ &= \sum_{j=0}^{k} q^{\binom{j}{2}} \left(q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q + q^k q^{\binom{j-1}{2}} \begin{bmatrix} k \\ j-1 \end{bmatrix}_q \right) (-1)^j U_{n+k+1-j}(z,m) \\ &= \sum_{j=0}^{k} q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+k+1-j}(z,m)\end{aligned}$$

and

$$\begin{aligned}&q^{m\binom{k+1}{2}} z^{k+1} U_{n-1-k}\left(q^{mk+m}z,m\right) \\ &= q^{m\binom{k}{2}} z^k \left(U_{n+1-k}\left(q^{mk-1}z,m\right) - U_{n-k}\left(q^{mk}z,m\right)\right) \\ &= q^{m\binom{k}{2}} q^k \sum_{j=0}^{k} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+k-j}\left(q^{j-1-k}z,m\right) - \\ &\quad \sum_{j=0}^{k} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+k-j}\left(q^{j-k}z,m\right) \\ &= \sum_{j=0}^{k} \left(q^k q^{\binom{k-j}{2}} \begin{bmatrix} k \\ k-j \end{bmatrix}_q + q^{\binom{k-j+1}{2}} \begin{bmatrix} k \\ k-j+1 \end{bmatrix}_q \right) (-1)^j U_{n+k-j}\left(q^{j-1-k}z,m\right) \\ &= \sum_{j=0}^{k} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q (-1)^j U_{n+k+1-j}\left(q^{j-1-k}z,m\right)\end{aligned}$$

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Lemma 7 *For every integer $n \geq 1$, one has*

$$\begin{aligned} & \mathbf{F}_{2n+1}\left(\frac{z}{q}, m\right) \\ = & (-z)^n q^{(m+1)\binom{n}{2}} + \sum_{j=0}^{n-1} q^{(m+1)\binom{j}{2}} (-z)^j \mathbf{L}_{2(n-j)}\left(q^{mj+j} z, m\right), \\ & \mathbf{F}_{2n+1}(z, m) \\ = & (-z)^n q^{\binom{n+1}{2}+m\binom{n}{2}} + \sum_{j=0}^{n-1} q^{j(2n-1)+(m-3)\binom{j}{2}} (-z)^j \mathbb{L}_{2(n-j)}\left(q^{mj-j} z, m\right). \end{aligned}$$

Proof. We use induction over n, the case $n = 1$ is given by the following relations, see [5]

$$\begin{aligned} \mathbf{L}_n(z, m) &= \mathbf{F}_{n+1}\left(\frac{z}{q}, m\right) + z\mathbf{F}_{n-1}(q^m z, m), \\ \mathbb{L}_n(z, m) &= \mathbf{F}_{n+1}(z, m) + q^{n-1} z\mathbf{F}_{n-1}\left(q^{m-1} z, m\right). \end{aligned}$$

We suppose the identities true for n, then

$$\begin{aligned} & \mathbf{F}_{2n+3}\left(\frac{z}{q}, m\right) \\ = & \mathbf{L}_{2n+2}(z, m) - z\mathbf{F}_{2n+1}(q^m z, m), \\ = & \mathbf{L}_{2n+2}(z, m) + (-z)^{n+1} q^{(m+1)\binom{n+1}{2}} - \\ & z\sum_{j=0}^{n-1} q^{\binom{j}{2}} \left(-q^{m+1} z\right)^j \mathbf{L}_{2(n-j)}\left(q^{mj+j+m+1} z, m\right), \\ = & (-z)^{n+1} q^{(m+1)\binom{n+1}{2}} + \sum_{j=0}^{n} q^{\binom{j}{2}} (-z)^j \mathbf{L}_{2(n+1-j)}\left(q^{mj+j} z, m\right). \end{aligned}$$

and

$$\begin{aligned} & \mathbf{F}_{2n+3}(z, m) \\ = & \mathbb{L}_{2n+2}(z, m) - q^{2n+1} z\mathbf{F}_{2n+1}\left(q^{m-1} z, m\right), \\ = & \mathbb{L}_{2n+2}(z, m) - q^{2n+1} z(-z)^n q^{\binom{n}{2}+m\binom{n+1}{2}} + \\ & q^{2n+1} z\sum_{j=0}^{n-1} q^{j(2n+1)+(m-3)\binom{j+1}{2}} (-z)^j \mathbb{L}_{2(n-j)}\left(q^{mj+m-1-j} z, m\right), \\ = & \mathbb{L}_{2n+2}(z, m) + (-z)^{n+1} q^{\binom{n+2}{2}m\binom{n+1}{2}} - \\ & q^{2n+1}\sum_{j=0}^{n-1} q^{j(2n+1)+(m-3)\binom{j+1}{2}} (-z)^j \mathbb{L}_{2(n-j)}\left(q^{mj+m-1-j} z, m\right), \\ = & (-z)^{n+1} q^{\binom{n+2}{2}m\binom{n+1}{2}} + \sum_{j=0}^{n} q^{j(2n+1)-(m-3)\binom{j}{2}} (-z)^j \mathbb{L}_{2(n+1-j)}\left(q^{mj-j} z, m\right). \end{aligned}$$

■

Proof of relations (13) and (14).. Replacing $U_n(z)$ by $\mathbf{L}_n(z)$ and $\mathbb{L}_n(z)$ respectively, in Lemma 6, we get

$$\begin{aligned}\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \mathbf{L}_{2n-k}(z,m) &= q^{(m+1)\binom{n}{2}} z^n 2, \\ \sum_{k=0}^{j} q^{\binom{k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k \mathbf{L}_{2n-k}\left(q^{2j}z,m\right) &= q^{(m+1)\binom{j}{2}+2nj} z^j \mathbf{L}_{2(n-j)}\left(q^{mj+j}z,m\right).\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \mathbb{L}_{2n-k}\left(q^{k-n}z,m\right) &= q^{m\binom{n}{2}} z^n 2, \\ \sum_{k=0}^{j} q^{\binom{j-k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k \mathbb{L}_{2n-k}\left(q^{k-2j}z,m\right) &= q^{(m-2)\binom{j}{2}-j} z^j \mathbb{L}_{2(n-j)}\left(q^{mj-j}z,m\right).\end{aligned}$$

using these relations in Lemma 7, we draw

$$\begin{aligned}2\mathbf{F}_{2n+1}\left(\frac{z}{q},m\right) &= \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n+k} \mathbf{L}_{2n-k}(z,m) + \\ &\quad 2\sum_{j=0}^{n-1}\sum_{k=0}^{j} \left(-q^{-2n}\right)^j q^{\binom{k}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^k \mathbf{L}_{2n-k}\left(q^{2j}z,m\right),\end{aligned}$$

and

$$\begin{aligned}2\mathbf{F}_{2n+1}(z,m) &= \sum_{k=0}^{n} q^{\binom{n-k}{2}+\binom{n+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n+k} \mathbb{L}_{2n-k}\left(q^{k-n}z,m\right) + \\ &\quad \sum_{j=0}^{n-1}\sum_{k=0}^{j} q^{2nj} q^{\binom{j-k}{2}-\binom{j}{2}} \begin{bmatrix} j \\ k \end{bmatrix}_q (-1)^{k+j} \mathbb{L}_{2n-k}\left(q^{k-2j}z,m\right)\end{aligned}$$

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Proof of relations (15) and (16).. For $k=n$ and $U_n(z)=\mathbf{F}_n(z)$ in Lemma 6, we have

$$\begin{aligned}\sum_{j=0}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j \mathbf{F}_{2n-j}(z,m) &= q^{m\binom{k}{2}+\binom{n}{2}} z^n \mathbf{F}_0\left(q^{mn-n}z,m\right) = 0, \\ \sum_{j=0}^{n} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j \mathbf{F}_{2n-j}\left(q^{j-n}z,m\right) &= q^{m\binom{n}{2}} z^n \mathbf{F}_0\left(q^{mn}z,m\right) = 0.\end{aligned}$$

■

Proof of relations (17), (18), (19) and (20).. It suffices to replace relations (15) and (16) in

$$\begin{aligned}\mathbf{L}_{2n-1}(z,m) &= 2\mathbf{F}_{2n}\left(\frac{z}{q},m\right) - \mathbf{F}_{2n-1}(z,m), \\ \mathbb{L}_{2n-1}(z,m) &= 2\mathbf{F}_{2n}(z,m) - \mathbf{F}_{2n-1}(z,m).\end{aligned}$$

■

Proof of relations (21) and (22).. According to Lemma 6, we have

$$\begin{aligned}\sum_{k=0}^{n-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k \mathbf{L}_{2n-2-k}(z,m) &= 2(z)^{n-1} q^{m\binom{n-1}{2}+\binom{n-1}{2}}, \\ \sum_{k=0}^{n-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k \mathbf{L}_{2n-1-k}(z,m) &= (z)^{n-1} q^{m\binom{n-1}{2}+\binom{n}{2}},\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n-1} q^{\binom{n-1-k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k \mathbb{L}_{2n-2-k}\left(q^{k+2-n}z,m\right) &= 2q^{m\binom{n-1}{2}} (qz)^{n-1} \\ \sum_{k=0}^{n-1} q^{\binom{n-1-k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k \mathbb{L}_{2n-1-k}\left(q^{k+1-n}z,m\right) &= q^{m\binom{n-1}{2}} z^{n-1}\end{aligned}$$

■

Then

$$\begin{aligned}&2\mathbf{L}_{2n-1}(z,m) \\ &= q^{n-1} \sum_{k=1}^{n} q^{\binom{k-1}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^{k+1} \mathbf{L}_{2n-1-k}(z,m) - \\ &\quad 2\sum_{k=1}^{n-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k \mathbf{L}_{2n-1-k}(z,m), \\ &= \sum_{k=1}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-q)^{k+1} \left(q^{n-k}\frac{[k]_q}{[n]_q} + 2\frac{[n-k]_q}{[n]_q} \right) \mathbf{L}_{2n-1-k}(z,m), \\ &= \sum_{k=1}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-q)^{k+1} \left(1 + \frac{[n-k]_q}{[n]_q} \right) \mathbf{L}_{2n-2-k}(z,m).\end{aligned}$$

and

$$\begin{aligned}&2\mathbb{L}_{2n-1}\left(q^{1-n}z,m\right) \\ &= q^{1-n} \sum_{k=1}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q (-1)^{k+1} \mathbb{L}_{2n-1-k}\left(q^{k+1-n}z,m\right) - \\ &\quad 2\sum_{k=1}^{n-1} q^{\binom{n-1-k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k \mathbb{L}_{2n-1-k}\left(q^{k+1-n}z,m\right), \\ &= \sum_{k=1}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k+1} q^{1-n} \left(\frac{[k]_q}{[n]_q} + 2q^k\frac{[n-k]_q}{[n]_q} \right) \mathbb{L}_{2n-1-k}\left(q^{k+1-n}z,m\right), \\ &= \sum_{k=1}^{n} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{k+1} q^{1-n} \left(1 + q^k\frac{[n-k]_q}{[k]_q} \right) \mathbb{L}_{2n-1-k}\left(q^{k+1-n}z,m\right).\end{aligned}$$

Proof of the relations (23), (24).. Using relations (13), (14), (21), (22) in

$$\begin{aligned}2\mathbf{F}_{2n}\left(\frac{z}{q},m\right) &= \mathbf{L}_{2n-1}(z,m) + \mathbf{F}_{2n-1}(z,m), \\ 2\mathbf{F}_{2n}(z,m) &= \mathbb{L}_{2n-1}(z,m) + \mathbf{F}_{2n-1}(z,m).\end{aligned}$$

we draw the results. ■

Remark 2 *Considering these results, we obtain a duality between the q-Lucas polynomials of the first kind and the q-Lucas polynomials of the second kind.*

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