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NP-hardness of the two-machine flowshop problem with coupled-operations

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Abstract: We consider the problem of the coupled operations on two machines with the objective to minimizing the makespan such as each job consists of two operations on the first machine separated by a time interval and only one operation on the second machine. We study the complexity of two special cases of the general problem and we show that it's NP-hard.

Keywords: coupled tasks, time lag, flow shop, makespan.

Résumé : Le problème étudié est flowshop à deux machines avec des opérations couplées sur la première machine séparées par un délai exact et d'une seule opération sur la deuxième machine. Notre objectif est de minimiser la date de fin de traitement. Nous montrons la NP-difficulté de deux de ses sous problèmes.

Mots clés : tâches couplées, temps de latence, atelier à cheminement unique, makespan.

1 Introduction

Scheduling of coupled-task problem was introduced by Shapiro in [16]. The problem consists of a set of n jobs, to be scheduled on a single machine. Each job j consists of two operations, these operations have to be executed in specified order with an intermediate exact delay L_j after the completion of the first operation. Each job j is described by a triplet (a_j, L_j, b_j) where a_j and b_j represents the processing times of the first and the second operation of jobs j, respectively. During the delay time L_j the machine is inactive and another job can be processed during the time rely L_j, as described in figure 5. The objective is determine the sequencing of operations of each job that minimize makespan or other regular objective function. According to the notation introduced by Graham et al [11], the problem of coupled-task with one machine with the objective of minimizing C_{max} noted by $1/Coup - Task, a_j, l_j, b_j/C_{max}$.

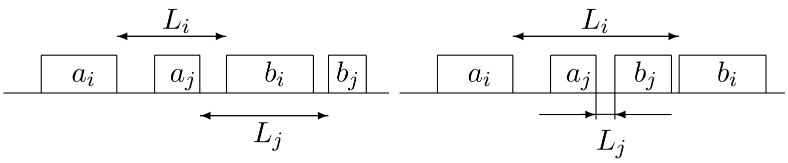

Figure 1: Examples of jobs interleaving

The motivation of a coupled-task problem stems from a scheduling problem of radar tasks which consists in the emission of the pulses and the reception of answers after the time interval, [16]. Shapiro [16], discussed the practical cases of the problem coupled-task and gave heuristics with numerical experiments. Orman and Potts [15] studied the single machine problem, with the objective of minimizing the C_{max}. They counted several problems and they have arranged them hierarchically according to their complexity. The problem $1/Coup - Task, a_j = a, L_j = l, b_j = b/C_{max}$ was left open by Orman and Potts [15] and for which others authors were interested. Ahr et al [3] proposed an exact algorithm using the dynamic programming which allows to resolve the problem for small instances where L is fixed. This algorithm was adapted by Brauner et al in [5] to resolve a coupled-task problem motivated by the time management problems of cyclic production with robots. Other researchers headed to the approximability of these problems. Thus, Ageev and Baburin [1] proposes an 7/4-approximation to solve the problem $1/Coup-Task, a_j = b_j = 1, L_j/C_{max}$. For the problems $1/Coup - Task, a_j, L_j, b_j/C_{max}$, Ageev and Kononov [2] gives several results of approximability and the limits of non-approximability according to the values of a_j and b_j. Few works were considered by adding constraints to the coupled tasks problem. Blazewicz et al [4] proved that the polynomial problem $1/Coup-task, a_j = b_j = 1, L_j = l/C_{max}$ is NP-hard by adding precedence constraints between the coupled tasks. Yu et al [18] proved that the problem on one machine $1/Coup - Task, a_j = b_j = 1, L_j/C_{max}$ is NP-hard. Simonin et al. [17] studied the problem of coupled-task with precedence constraints. They proposed a polynomial algorithm for this special problem.

Scheduling problem with exact delay is also considered in the context of flowshop environment, mainly in the two machines flowshop. The two machines flowshop scheduling problem with exact delay consists of n jobs, each job j consists of two operations $O_{A,j}$

and $O_{B,j}$ that will be processed on machines A and B, respectively. Operation $O_{B,j}$ is started on machine B after exact time delay L_j of completing of operation $O_{A,j}$. This problem is denoted $F2/L_j/C_{max}$ in the literature. Yu et al. [18] showed that problem $F2/a_j = b_j = 1, L_j/C_{max}$ is NP-hard, Ageev and Baburin [1] proposes an 3/2-approximation to solve the problem $F2/a_j = b_j = 1, L_j/C_{max}$. Ageev and Kononov [2] gives several results of approximability and the limits of non-approximability for problem $F2/L_j/C_{max}$ according to the values of a_j and b_j. Flowshop scheduling with exact time delay is special case of general problem in which time delay is bounded by minimal L_j^{min} and maximal L_j^{max} time delay. Mitten [14] provides a polynomial algorithm to minimize the makespan in two machines permutation flow shop scheduling problem with minimal time delay, i.e., $L_j^{max} = +\infty$. Lenstra et al. [13], studied the general two machines flow shop scheduling problem with minimal delay $F2/L_j^{min}/C_{max}$ and they proved that this problem is NP-hard in the strong senses. Dell Amico [6], focuses on the makespan minimization in two-machine flowshop with minimal time delay and presents a tabu search approach. Other research works related to flowshop scheduling with minimal and/or maximal time delay are studied in, [8], [9].

In this paper we consider the two-machine flowshop scheduling problem with coupled-operations. This problem consists of scheduling a set of jobs, each job is composed of two coupled-operations that should be processed on the first and on the second machine. The process of the second coupled-operation on the second machine starts only after the completion time of the first coupled-operation on the first machine. There are two other models, namely a model 1 whose the coupled-operations are carried out only on the first machine and model 2 whose couple-operations are processed on the second machine. Note that for the makespan minimization, model 1 and model 2 are equivalent. In this paper, our study is focused on model 1 in order to minimize the makespan. To our the best of knowledge this problem never considered in the literature. The motivation of flowshop scheduling problem with coupled-operations appears in workshops chemical productions where one machine must carry out several operations of the same job and an exact delay is imposed between the execution of each two consecutive operations due to the chemical reactions.

2 Problem formulation and classification

We consider the flowshop scheduling problem with two machines M_1 and M_2. A set $J = \{1, \ldots, n\}$ of jobs need to be scheduled on both machines. Each job j comprises a coupled-operation $O_{1,j}$ and a operation $O_{2,j}$ to be processed on machines M_1 and M_2, respectively, in this order for all jobs. Each coupled-operation $O_{1,j}$ of job j is described by a triplet (a_j, L_j, b_j) where a_j and b_j represents the processing times of the first and the second operation of coupled-operation $O_{1,j}$, respectively, whereas the operation $O_{2,j}$ is described by it processing time c_j. For each job j, operation $O_{2,j}$ can starts only if the coupled-operation $O_{1,j}$ is completed. The objective is to determine the sequencing of jobs that minimize the makespan.

In [15], Orman and potts studied the problem of the coupled-tasks on a single machine and they derive several results of problems depending of values of a_j, L_j and b_j, furthermore, they provide a classification of these problems depending on their complexity. Clearly, all NP-hard problems on a signal machine, remain NP-Hard in case of flowshop environment. Thus, in this paper, we focus our studies on problems that already proved polynomial in case of single machine. Figure 2 provide the list of all polynomial problems proposed by Orman and Potts [15] and the relation between these problems.

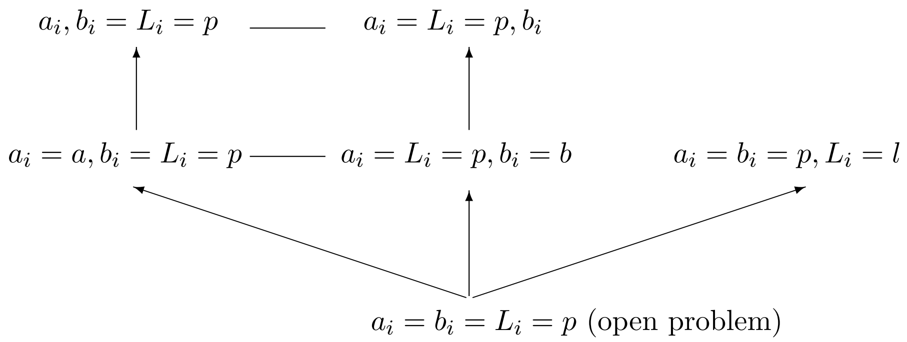

Figure 2: Polynomial problems of Orman and Potts

The problem of two-machine flow shop with coupled-operations on the first machine has not been tackled in the literature, then we are interested in this type of problem based on the polynomial problems class to define problems to be studied. However, by adding one machine to get two machines flow shop problem with coupled tasks on the first machine and one operation with processing time c_j on the second machine, we obtain the various problems presented in the following scheme.

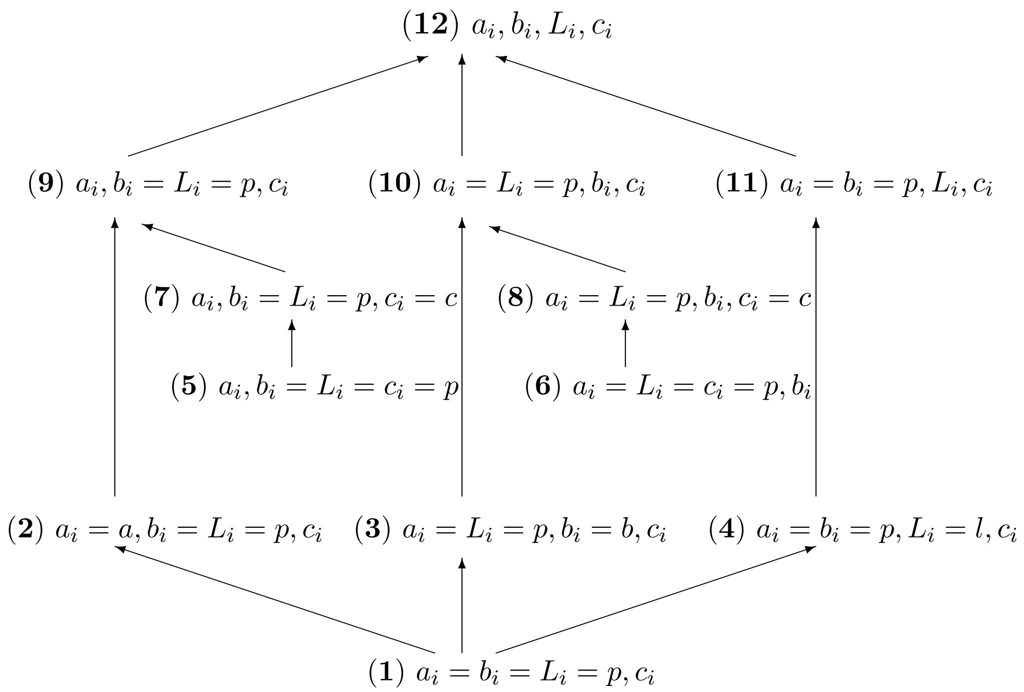

Figure 3: Problems classification

In this work, we present the complexity of the sub-problems (9) and (10) of the problem of the flow shop on two machines with coupled tasks on the first machine with the objective to minimize the makespan.

3 The first subproblem

In this section, we Consider the problem $F2/Coup-Opr(1), a_i, b_i = L_i = p, c_i/C_{max}$, abbreviated in the following as problem $F2C(a_i, b_i = L_i = p, c_i)$. We show that the problem $F2C(a_i, b_i = L_i = p, c_i)$ is NP-hard using a reduction of the partition problem with equal size, which is known to NP-Hard, [10].

The Partition with Equal Size problem (PES) is stated as follows : Given a set $E = \{e_1, e_2, ..., e_{2n}\}$ of $2n$ positives integers, where $\sum_{i=1}^{2n} e_i = 2B$ for some integer B. Does there exist a partition of E into two disjoint subsets E_1 and E_2 such as $\sum\limits_{i \in E_1} e_i = \sum\limits_{i \in E_2} e_i = B$ and $|E_1| = |E_2| = n$? This problem remains NP-hard even each $e_i > 1$. In our proof we assume that all $e_i > 1$.

Given an arbitrary instance of PES, we build an instance $\mathcal{I}$ of problem $F2C(a_i, b_i = L_i = p, c_i)$ with a set of $4n+2$ jobs as follows:

- Jobs of type U, denoted U_i, $i = 1, \dots, 2n$;
- n identical jobs denoted V;
- $n+1$ identical jobs denoted W;
- One job denoted T;

For all the jobs, we set $L_i = b_i = p$, $i = 1, \dots, 4n+2$ where $p > B$. Processing times of jobs on machine M_1 and M_2 are given in the Table 3.

Jobs	a_i	c_i
U_i, $i = 1, \dots, 2n$	$p - e_i$	e_i
V	$p+1$	$4p$
W	p	0
T	$B+p+1-n$	$(4n+1)p-2B$

Table 1: Jobs Processing times

Let the threshold for the makespan be y, where $y = 4(2n+1)p+1$.

In the following, the notation $(VW)-job$ refers to the composite job (VW) in which jobs V and W are interleaved and the first operation of V is at the first position. Furthermore, the composite (VW) can be seen as compact job with processing times $4p+1$ and $4p$ on machines M_1 and M_2, respectively, and has a time-lag $l = -p$, and can be scheduled as shown in Figure 4.

In an instance $\mathcal{I}$ of the scheduling problem $F2C(a_i, b_i = L_i = p, c_i)$, a schedule is said to be feasible if the makespan is lower or equal to y. The decision problem asks: is there a feasible schedule to the problem $F2C(a_i, b_i = L_i = p, c_i)$ with makespan less than or equal to y? We have the following results.

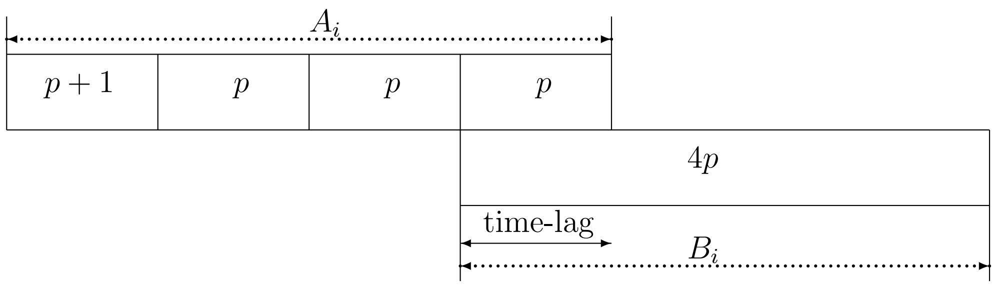

Figure 4: Example of composite job (VW)

Lemma 1 *Given an instance of the The Partition with Equal Size problem, if there is a solution to this instance, there exists a solution to the corresponding instance of the problem $F2C(a_i, b_i = L_i = p, c_i)$ with makespan less or equal to y.*

Proof. If there is a solution to an instance of The Partition with Equal Size problem, i.e., there is a partition E_1 and E_2 of E such that $\sum_{i \in E_1} e_i = \sum_{i \in E_2} e_i = B$ where E_1 and E_2 each contains exactly n elements, then we can schedule the jobs of the corresponding instance $\mathcal{I}$ as follows.

Let J_1 and J_2 be the subset of the $U-$Jobs corresponding to the subsets E_1 and E_2, respectively. Then the desired schedule S exists where the completion time $C_{max}(S)$ of schedule S is equal to $4(2n+1)p+1$. In this schedule, the $U-$Jobs of J_1 (J_2) are interleaved with $V-$Jobs ($W-$Jobs), and one job W is interleaved with job T. Let $(VU)-$Jobs $((VU)_1, \ldots, (VU)_n)$, $(WU)-$Jobs $((WU)_1, \ldots, (WU)_n)$ and $(WT)-$job be the composite jobs obtained by the interleaving operation. Schedule S is constructed as follows (see Figure 6): start by scheduling $(VU)-$jobs, followed by $(TW)-$job then finish with $(WU)-$jobs. The order of composite jobs of set $(VU)-$Jobs and $(WU)-$Jobs in schedule S can be chosen in any order. Figure 6 pictures the resulting schedule. In schedule S there is no idle-time between composite jobs on machine M_1 and M_2, then,

$$C_{max}(S) = p+1+2p+4np+\sum_{i \in J_1} e_i + (4n+1)p - 2B + \sum_{i \in J_2} e_i = 4p+8np+1 = 4(2n+1)p+1.$$

■

In the following, we show that if there exists a solution to an instance of the scheduling problem with makespan less or equal to y, then there exists a solution to the corresponding instance of the Partition with Equal Size problem. We show that for any feasible schedule S, the minimum value of makespan is y. Furthermore, to obtain a schedule with makespan equal to y, there must be a solution to the corresponding instance Partition with Equal Size problem. In order to show this result we need to establish the following.

1. All jobs are interleaved.
2. There is no composite (UU)-Jobs.
3. The $T-$job is interleaved with a $W-$Job.

Let X_1, X_2, Y_1, Y_2 be the processing times of interleaved jobs in relation to the position of the job T in schedule S as showed in Figure 5. Clearly, $LB = X_1 + \max\{X_2, Y_2\}$ is a lower bound for the makespan of schedule S.

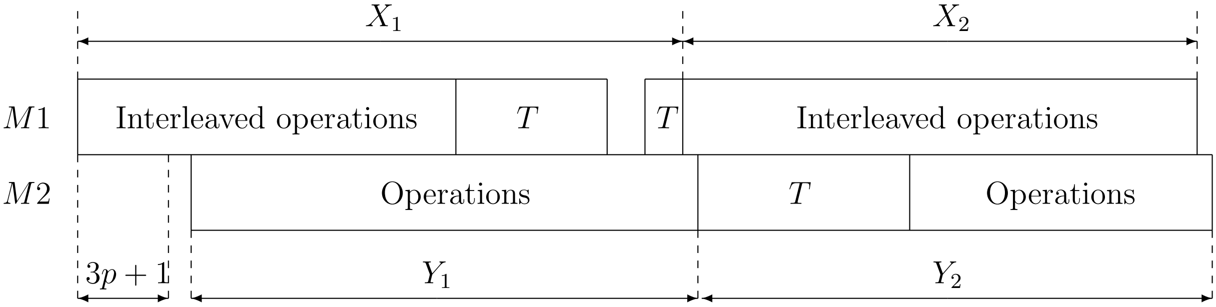

Figure 5: Some notations

Let A_i and B_i be the processing times of composite jobs on machines M_1 and M_2, respectively, and let l_i the time-lag of the composite jobs as showed in Figure 4. Table 2 summarizes the values of A_i, B_i and l_i of all possible composite jobs related our instance of scheduling problem. Furthermore, depending on the processing times of composite jobs, we have,

Jobs	A_i	B_i	l_i
(V, U_i)	$4p + 1$	$4p + e_i$	$-p$
(V, W)	$4p + 1$	$4p$	$-p$
(T, U_i)	$B + 4p + 1 - n$	$(4n + 1)p - 2B + e_i$	$-p$
(T, W)	$B + 4p + 1 - n$	$(4n + 1)p - 2B$	$-p$
(U_i, U_j)	$4p - e_i$	$e_i + e_j$	$-e_i$
(U_i, W)	$4p - e_i$	e_i	$-e_i$
(W, W)	$4p$	0	0

Table 2: Processing times of composite jobs.

- For any composite job, the jobs V and T can only be in first position and
- The composite job (U_iW) has small processing time than (WU_i) on the first machine, than it is easy to show that in schedule S, all composite job (WU_i) can be transformed into (U_iW) without increasing the value of makespan.

Lemma 2 *In feasible schedule S, All jobs are interleaved.*

Proof. Assume that in schedule S at least two jobs are not interleaved. Let consider that the $U-$jobs are indexed in non increasing order of e_j values of their processing time a_j. Since the total processing time of composite jobs on the first machine is a lower bound of the makespan, then the best interleaving operation of jobs that minimize the total processing time on the first machine is a lower bound of schedule S. Let consider the following interleaving operation of jobs: n composite jobs $((VW))$, one composite job (TW), and $n-1$ composite jobs (U_iU_k) where $i = n, \ldots, 2n-2$; $k = 1, \ldots n-1$, and jobs U_{2n-1} and U_{2n} are scheduled alone, respectively. It is easy to see that this interleaving operation of jobs provides the minimal total processing time of jobs on the first machine when exactly two jobs are not interleaved. Thus,

$$\begin{aligned} C_{max}(S) &\geq (B+p+1-n+3p)+(4np+n)+n.4p-\sum_{i=n-1}^{2n-2} e_i - e_{2n-1} - e_{2n} \\ &\geq 4(2n+1)p+1+2p+B-\sum_{i=n-1}^{2n} e_i = y+2p+B-\sum_{i=n-1}^{2n} e_i \end{aligned}$$

Since $p > B$ and $\sum_{i=n-1}^{2n} e_i < 2B$ then $2p+B-\sum_{i=n-1}^{2n} e_i > 0$, hence, $C_{max}(S) > y$. then in feasible solution all jobs are interleaved. ■

Lemma 3 *In feasible schedule S, there is no composite $(UU)-$Jobs.*

Proof. Let assume that there exists in schedule S' one composite $(UU)-$Jobs. Assume that this $(UU)-$Job is composed of a pair (U_i, U_j) of jobs in which U_i is interleaved with U_j and the other $U-$Jobs are interleaved with the other type, $V-$Jobs, $W-$Jobs and $T-$Job. We distinguish the following cases:

a. The job T is interleaved with a $W-$job, then it remains n $W-$Jobs, n $V-$Jobs and $(2n-2)$ $U-$Jobs. Depending on these remaining jobs, two ways of their interleaving are possible, namely,

   1. (TW), (WW), $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$, where $|I_1| = n$, $|I_2| = n-2$ and $I_1 \cup I_2 \cup \{i,j\} = \{1,\dots,2n\}$.
   2. (TW), (VW), $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$, where $|I_1| = n-1$, $|I_2| = n-1$ and $I_1 \cup I_2 \cup \{i,j\} = \{1,\dots,2n\}$.

b. The job T is interleaved with a $U-$job, then it remains n $V-$Jobs, $(n+1)$ $W-$Jobs and $(2n-3)$ $U-$Jobs. Again, depending on these remaining jobs, three ways of their interleaving are possible, namely,

   1. (TU_r), (WW), (WW), $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$, where $|I_1| = n$, $|I_2| = n-3$ and $I_1 \cup I_2 \cup \{i,j,r\} = \{1,\dots,2n\}$.
   2. (TU_r), (WW), (VW), $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$, where $|I_1| = n-1$, $|I_2| = n-2$ and $I_1 \cup I_2 \cup \{i,j,r\} = \{1,\dots,2n\}$.
   3. (T, U_r), (VW), (VW), $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$, where $|I_1| = n-2$, $|I_2| = n-1$ and $I_1 \cup I_2 \cup \{i,j,r\} = \{1,\dots,2n\}$.

In the following we examine the value of makespan of each sequence of interleaving jobs.

Case a.1: According to Mitten's algorithm [8], the optimal schedule of the composite jobs of the case 1.a is given by the sequence $S = \langle (VU_k)_{k\in I_1}, (TW), (U_iU_j), (U_kW)_{k\in I_2}, (WW)\rangle$. According to the processing times of these composite jobs given in Table 2, the parameters X_1, X_2, Y_1 and Y_2 (see figure 5) of the above order are as follow,

$$X_1 = 4np + 3p + B + 1, \quad X_2 = 4pn + p - (\sum_{k \in I_2} e_k + e_i).$$
$$Y_1 = 4np + (\sum_{k \in I_1} e_k), \qquad Y_2 = 4np + p - 2B + (\sum_{k \in I_2} e_k + e_i + e_j).$$

Then, the lower bound of $C_{max}(S)$ is

$$\begin{aligned} LB &= X_1 + \max\{X_2, Y_2\} = 8np + 4p + 1 + \max\{B - (\sum_{k \in I_2} e_k + e_i), (\sum_{k \in I_2} e_k + e_i + e_j) - B\} \\ &= y + \max\{B - (\sum_{k \in I_2} e_k + e_i), (\sum_{k \in I_2} e_k + e_i + e_j) - B\} \end{aligned}$$

Since $\max\{B - (\sum_{k \in I_2} e_k + e_i), (\sum_{k \in I_2} e_k + e_i + e_j) - B\} > 0$ then $LB > y$. Thus S is not a feasible solution.

Case a.2: The optimal schedule in this case is $S = \langle (VU_k)_{k \in I_1}, (TW), (VW), (U_iU_j), (U_kW)_{k \in I_2} \rangle$. The parameters X_1, X_2, Y_1 and Y_2 are as follow.

$$X_1 = 4np - p + B, \qquad X_2 = 4np + 5p + 1 - (\sum_{k \in I_2} e_k + e_i)$$
$$Y_1 = 4np - 4p + (\sum_{k \in I_1} e_k), \quad Y_2 = 4np + 5p - 2B + (\sum_{k \in I_2} e_k + e_i + e_j).$$

The value of lower bound is

$$\begin{aligned} LB &= X_1 + \max\{X_2, Y_2\} = 8np + 4p + 1 + \max\{B - (\sum_{k \in I_2} e_k + e_i), (\sum_{k \in I_2} e_k + e_i + e_j) - B - 1\}. \\ &= y + \max\{B - (\sum_{k \in I_2} e_k + e_i), (\sum_{k \in I_2} e_k + e_i + e_j) - B - 1\} \end{aligned}$$

Since $\forall k, e_k > 1$, we have $\max\{B - (\sum_{k \in I_2} e_k + e_i), (\sum_{k \in I_2} e_k + e_i + e_j) - B - 1\} > 0$, then $LB > y$. Thus S is not a feasible solution.

Case b.1: The optimal sequence of the composite jobs in this case is $S = \langle (VU_k)_{k \in I_1}, (TU_r),$ $(U_iU_j), (U_kW)_{k \in I_2}, (WW), (WW) \rangle$. The parameters X_1, X_2, Y_1 and Y_2 are,

$$X_1 = 4np + B + 3p + 1, \quad X_2 = 4np + p - (\sum_{k \in I_2} e_k + e_i)$$
$$Y_1 = 4np + \sum_{l \in k_1} e_k, \qquad Y_2 = 4np + p - 2B + (\sum_{k \in I_2} e_k + e_i + e_j + e_r).$$

Then the lowed bound of $C_{max}(S)$ is $LB = X_1 + \max\{X_2, Y_2\} = y + \max\{B - (\sum_{l \in I_2} e_l + e_i), (\sum_{l \in I_2} e_l + e_i + e_j + e_k) - B\}$. Since $\max\{B - (\sum_{l \in I_2} e_l + e_i), (\sum_{l \in I_2} e_l + e_i + e_j + e_k) - B\} > 0$ then $LB > y$. Thus, S is not a feasible solution.

Case b.2: The optimal sequence of the composite jobs in this case is $S = \langle (VU_k)_{k \in I_1}, (TU_r),$ $(VW), (U_iU_j), (U_kW)_{k \in I_2}, (WW) \rangle$ and the parameters X_1, X_2, Y_1 and Y_2 are

$$X_1 = 4np - p + B, \qquad X_2 = 4np + 5p - (\sum_{k\in I_2} e_k + e_i)$$
$$Y_1 = 4np - 4p + (\sum_{k\in I_1} e_k), \quad Y_2 = 4np + 5p - 2B + (\sum_{k\in I_2} e_k + e_i + e_r).$$

Then the lowed bound of $C_{max}(S)$ is $LB = X_1 + \max\{X_2, Y_2\} = y + \max\{B - (\sum_{k\in I_2} e_k + e_i + 1), (\sum_{k\in I_2} e_k + e_i + e_j + e_r) - B - 1\}$. Since $\forall i, e_i > 1$, then $\max\{B - (\sum_{k\in I_2} e_k + e_i + 1), (\sum_{k\in I_2} e_k + e_i + e_j + e_r) - B - 1\} > 0$, then $LB > y$. Thus, S is not a feasible solution.

Case b.3: The optimal sequence of the composite jobs of here is $S = \langle (VU_k)_{k\in I_1}, (TU_r), (VW), (VW), (U_iU_j), (U_kW)_{k\in I_2}$ and the parameters X_1, X_2, Y_1 and Y_2 are

$$X_1 = 4np - 5p - 1 + B, \qquad X_2 = 4np + 9p + 2 - (\sum_{k\in I_2} e_k + e_i)$$
$$Y_1 = 4np - 8p + (\sum_{k\in I_1} e_k), \quad Y_2 = 4np + 9p - 2B + (\sum_{k\in I_2} e_k + e_i + e_j + e_r).$$

Then the lowed bound of $C_{max}(S)$ is $LB = X_1 + \max\{X_2, Y_2\} = y + \max\{B - (\sum_{k\in I_2} e_k + e_i), (\sum_{k\in I_2} e_k + e_i + e_j + e_r) - B - 1\}$. Since $\forall i, e_i > 1$, we have $\max\{B - (\sum_{k\in I_2} e_k + e_i), (\sum_{k\in I_2} e_k + e_i + e_j + e_r) - B - 1\} > 0$, then $LB > y$. Thus S is not a feasible solution. Note that if we have more than one interleaving $UU-$jobs then X_2 increases and $LB > y$. ■

Lemma 4 *In a feasible schedule S, the $T-$Job is interleaved with $W-$Job.*

Proof. Let assume that there exists a schedule S' in which T is interleaved with $U-$job. Let U_r the $U-$job interleaved with T. From lemmas 2 and 3, the only possible composite jobs are

1. $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$ where $|I_1| = n-2$, $|I_2| = n+1$ and $I_1 \cup I_2 \cup \{r\} = \{1, \ldots, 2n\}$.
2. $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$, (VW) where $|I_1| = n - 1$, $|I_2| = n$ and $I_1 \cup I_2 \cup \{r\} = \{1, \ldots, 2n\}$.
3. $(VU_k)_{k\in I_1}$, $(U_kW)_{k\in I_2}$, (WW) where $|I_1| = n$, $|I_2| = n - 1$ and $I_1 \cup I_2 \cup \{r\} = \{1, \ldots, 2n\}$.

For above cases the optimal sequences are

- Case 1. $S' = \langle (VU_k)_{k\in I_1}, (TU_r), (U_kW)_{k\in I_2}\rangle$
- Case 2. $S' = \langle (VU_k)_{k\in I_1}, (TU_r), (VW), (U_kW)_{k\in I_2}\rangle$
- Case 3. $S' = \langle (VU_k)_{k\in I_1}, (TU_r), (U_kW)_{k\in I_2}, (WW)\rangle$

Similarly to the proof of lemma 3, it is easy to show that for each above case $C_{max}(S') > y$, then in schedule S job R is interleaved with job W. ■

Lemma 5 *If there is a solution to an instance of problem $F2C(a_i, b_i = L_i = p, c_i)$, then there exists a solution to the corresponding instance of the Partition with Equal Size problem.*

Proof. From lemmas 2, 3 and 4, the unique interleaved operations of jobs in schedule S is $(VU_k)_{k\in I_1}$, (TW) and $(U_kW)_{k\in I_2}$ where where $|I_1| = n$, $|I_2| = n$ and $I_1 \cup I_2 = \{1, \ldots, 2n\}$. The optimal sequence of these composite jobs is $S' = \langle (VU_k)_{k\in I_1}, (TW), (U_kW)_{k\in I_2}\rangle$. The parameters X_1, Y_1, X_2 and Y_2 of this sequence are,

$$X_1 = 4np + p + 1 + B, \quad X_2 = 4np + p - \sum_{k\in I_2} e_k$$
$$Y_1 = 4np + \sum_{k\in I_1} e_k, \qquad Y_2 = 4np + p - 2B + \sum_{k\in I_2} e_k.$$

The lowed bound of $C_{max}(S)$ is $LB = X_1 + \max\{X_2, Y_2\} = y + \max\{B - \sum_{k\in I_2} e_k, \sum_{k\in I_2} e_k - B\}$. since $C_{max}(S) \le y$, then $\max\{B - \sum_{k\in I_2} e_k, \sum_{k\in I_2} e_k - B\} = 0$. Thus $\sum_{k\in I_2} e_k = B$ and $\sum_{k\in I_1} e_k = B$. since $|I_1| = |I_2| = n$, then we obtain a solution for the PES problem. ■

From Lemmas 1 and 5, we know that there is a solution to the instance of the Partition with Equal Size problem if and only if there is a solution to the corresponding instance $\mathcal{I}$ of the scheduling problem. Therefore, the following result.

Theorem 6 *The problem $F2C(a_i, b_i = L_i = p, c_i)$ is binary NP-hard.*

4 The second subproblem

In this section, we show that the problem $F2/Coup - Opr(1), a_i = L_i = p, b_i, c_i/C_{max}$, abbreviated in the following as problem $F2C(a_i = L_i = p, b_i, c_i)$ is NP-hard using a reduction of the Partition problem with Equal Size used in section **??**.

Given an arbitrary instance of the Partition problem with Equal Size used in section **??**, we build an instance $(\mathcal{I})$ of problem $F2C(a_i = L_i = p, b_i, c_i)$ with a set of $4n + 4$ jobs as follows:

- Jobs of type U, denoted U_i, $i = 1, \ldots, 2n$;
- n identical jobs denoted V;
- $n + 2$ identical jobs denoted W;
- One job denoted T;
- One job denoted R;

For all the jobs, we set $a_i = L_i = p$, $i = 1, \ldots, 4n + 4$ where $p > B$. Processing times of jobs on machine M_1 and M_2 are given in the table 3.

Let the threshold for the makespan be y, where $y = 9p(n + 1)p + n + 2$. We have the following result.

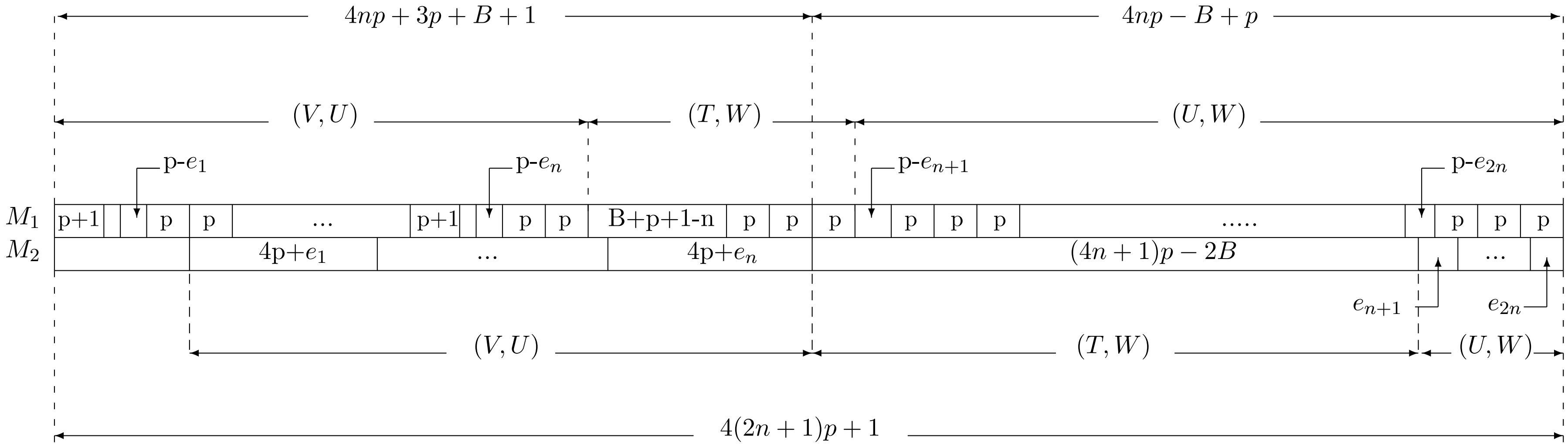

Figure 6: Schedule S

Jobs	b_i	c_i
U_i, $i = 1, \ldots, 2n$	$p - e_i$	e_i
V	$p + 1$	$5p + 1$
W	p	0
T	$(n+2)p + B + 1$	$4p(n+1) - 2B + 1$
R	$p + 1$	0

Table 3: Jobs Processing times

Lemma 7 *Given an instance of the the Partition with Equal Size problem, if there is a solution to this instance, there exists a solution to the corresponding instance of the problem* $F2C(a_i = L_i = p, b_i, c_i)$ *with makespan less or equal to* y.

Proof. Assume that the Partition with Equal Size problem problem has a solution, and let E_1 and E_2 be the required subset of E such that $\sum_{i \in E_1} e_i = \sum_{i \in E_2} e_i = B$ and $|E_1| = |E_2| = n$. Let J_1 and J_2 be the subset of the $U-$Jobs corresponding to the subsets E_1 and E_2, respectively. Then the desired schedule S exists where the completion time $C_{max}(S)$ of schedule S is equal to $9p(n+1)p + n + 2$. The composite jobs and their schedule is given by sequence $S = \langle (U_k V)_{k \in J_1}, (WT), (WU_k)_{k \in J_2}, (WR) \rangle$. ■

Assume now that the there exists a solution to an instance of the scheduling problem with makespan less or equal to y, then we show that there exists a solution to the corresponding instance of the Partition with Equal Size problem. In order to show this result we need to establish the following.

1. All jobs are interleaved.
2. There is no composite (UU)-Jobs.
3. The $T-$job is interleaved with a $W-$Job.
4. The $R-$job is interleaved with a $W-$Job

As presented in section **??** Table 4 summarizes the values of A_i, B_i and l_i of all possible composite jobs related to the instance of scheduling problem.

The following results establish the statement (1)-(4).

Lemma 8 *In a feasible schedule* S, *All jobs are interleaved.*

Proof. The proof is similar to the proof of lemma 2 ■

Lemma 9 *In a feasible schedule* S, *there is no composite* $(UU)-$*Jobs.*

Jobs	A_i	B_i	l_i
(U_i, V)	$4p+1$	$5p+1+e_i$	$-e_i$
(W, V)	$4p+1$	$5p+1$	0
(W, U_i)	$4p-e_i$	e_i	0
(U_i, R)	$4p+1$	e_i	$-e_i$
(W, W)	$4p$	0	0
(W, R)	$4p+1$	0	0
(U_i, U_j)	$4p-e_i$	e_i+e_j	$-e_i$
(U_i, T)	$B+(n+5)p+1$	$4p(n+1)-2B+e_i$	$-e_i$
(W, T)	$B+(n+5)p+1$	$4p(n+1)-2B$	0

Table 4: Processing times of composite jobs.

Proof. The proof is similar to the proof of lemma 3 ∎

Lemma 10 *In a feasible schedule S, the $T-$Job is interleaved with $W-$Job.*

Proof. The proof is similar to the proof of lemma 4 ∎

Lemma 11 *In a feasible schedule S, the $R-$Job is interleaved with $W-$Job.*

Proof. Let assume that there exists a schedule S' in which R is interleaved with $U-$job. Let U_r be the $U-$job interleaved with R. From lemmas 8, 9 and 10, the only possible composite jobs are

1. (WT), (U_rR), $(U_kV)_{k\in I_1}$, $(WU_k)_{k\in I_2}$, and (WW) where $|I_1| = n$, $|I_2| = n-1$ and $I_1 \cup I_2 \cup \{r\} = \{1, \ldots, 2n\}$.
2. (WT), (U_rR), (WV), $(VU_k)_{k\in I_1}$, and $(U_kW)_{k\in I_2}$, where $|I_1| = n-1$, $|I_2| = n$ and $I_1 \cup I_2 \cup \{r\} = \{1, \ldots, 2n\}$.

For above cases the optimal sequences are

- Case 1. $S' = \langle (VU_k)_{k\in I_1}, (WT), (U_kW)_{k\in I_2}, (U_rR), (WW)\rangle$
- Case 2. $S' = \langle (VU_k)_{k\in I_1}, (WV), (WT), (U_kW)_{k\in I_2}, (U_rR)\rangle$

Similarly to the proof of lemma 3, it is easy to show that for each above case $C_{max}(S') > y$, then in schedule S job R is interleaved with job W. ∎

Lemma 12 *If there is a solution to an instance of problem $F2C(a_i = L_i = p, b_i, c_i)$, then there exists a solution to the corresponding instance of the Partition with Equal Size problem.*

Proof. From lemmas 8, 9, 10 and 11, there unique interleaved operations of jobs in schedule S is (WT), WR, $(U_kV)_{k\in I_1}$, $(WU_k)_{k\in I_2}$ where $|I_1| = n$, $|I_2| = n$ and $I_1 \cup I_2 = \{1, \ldots, 2n\}$ The optimal sequence of these composite jobs is $S' = \langle (U_kV)_{k\in I_1}$, (WT), $(WU_k)_{k\in I_2}(WR)\rangle$. The parameters X_1, X_2 and Y_2 of this sequence are,

$$X_1 = 5np + 5p + B + n + 1, \qquad X_2 = 4np + 4p + 1 - \sum_{k\in I_2} e_k$$
$$Y_2 = 4np + 4p + 1 - 2B + \sum_{k\in I_2} e_k.$$

The lowed bound of $C_{max}(S)$ is $LB = X_1 + \max\{X_2, Y_2\} = y + \max\{B - \sum_{k\in I_2} e_k, \sum_{k\in I_2} e_k - B\}$. since $C_{max}(S) \leq y$, then $\max\{B - \sum_{k\in I_2} e_k, \sum_{k\in I_2} e_k - B\} = 0$. Thus $\sum_{k\in I_2} e_k = B$ and $\sum_{k\in I_1} e_k = B$. since $|I_1| = |I_2| = n$, then we obtain a solution for the PES problem. ■

From Lemmas 7 and 12, we know that there is a solution to the instance of the Partition with Equal Size problem if and only if there is a solution to the corresponding instance $\mathcal{I}$ of the scheduling problem. Therefore, the following result.

Theorem 13 *The problem $F2C(a_i = L_i = p, b_i, c_i)$ is binary NP-hard.*

5 Conclusion

In this article, we studied the complexity of flow shop problem with coupled tasks. Our problem consists in a flow shop with two machines with coupled tasks on the first machine such as each job consists in two operations on the first machine separated by a time lag and one operation on the second machine in order to minimize the total completion time. The problem is NP-hard in its general form. We have shown that two of its sub-problems are NP-hard.

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