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A combinatorial contribution to the multinomial Chu-Vandermonde convolution

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Abstract: A combinatorial proof to multinomial Chu-Vandermonde convolution is given with an extension to polynomial case. We deal also with some probabilistic contributions as a simple application to random matrices.

Keywords: Chu-Vandermonde convolution; Hypergeometric distribution probability; Random matrix

Résumé : Une preuve combinatoire pour la convolution multinomial de Chu-Vandermonde est donné avec une extension au cas polynomiale. Nous donnons aussi une contributions probabilistes comme une application simple aux matrices aléatoires.

Mots clés : Convolution de Chu-Vandermonde; distribution hypergéométrique; matrice aléatoire

1 Introduction

Chu-Vandermonde identity states that for all $m, n, r \in \mathbb{N}$, we have the following

$$\binom{n+m}{r} = \sum_k \binom{n}{k}\binom{m}{r-k}, \tag{1}$$

where $\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{for } \ 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$

$\binom{n}{k}$ is the binomial coefficient, combinatorially it counts the number of ways to take k members from n candidates.

Relation (1) admits the following well known extension: given $s \in \mathbb{N}$ and $n_1, n_2, \ldots, n_s$, r nonnegative integers, the following identity holds

$$\binom{n_1+n_2+\cdots+n_s}{r} = \sum_{k_1+k_2+\cdots+k_s=r} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}. \tag{2}$$

Divided by the left expression of both sides of identities (1) and (2) respectively, the summands terms are interpreted as the hypergeometric and the polyhypergeometric probability distributions.

Our aim is to give some extensions of Chu-Vandermonde identity to the multinomial case. This work completes those of Gould [2, 3].

2 Multinomial Chu-Vandermonde identity

We use the following notation for the multinomial coefficient, for all $n_1, n_2, \ldots, n_t$ and $n \in \mathbb{Z}$,

$$\binom{n}{n_1, n_2, \ldots, n_t} = \begin{cases} \frac{n!}{n_1!n_2!\cdots n_t!} & \text{if } n_1, n_2, \ldots, n_t \text{ ares integers with sum } n, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The main advantages of such an interpretation of multinomial coefficient is that one can omit the use of exact limit in sums like $\sum_{\substack{n_1,n_2,\ldots,n_t/ \\ n_1+n_2+\cdots+n_t=n}} \binom{n}{n_1,n_2,\ldots,n_t}$ by simply writing $\sum_{n_1,n_2,\ldots,n_t} \binom{n}{n_1,n_2,\ldots,n_t}$ instead. In the sequel, for the sake of convenience, we exploit this kind of allowance.

The multinomial coefficient $\binom{n}{n_1,n_2,\ldots,n_t}$ counts the number of ways to constitute t distinguishable committees from n candidates such that the first committee contains n_1 indistinguishable members, the second contains n_2 indistinguishable members, ... and the t^{th} contains n_t indistinguishable members.

Theorem 1 *Let $t \in \mathbb{N}$ and $r_1, r_2, \ldots, r_t, n, m$ be nonnegative integers, the following identity holds*

$$\binom{n+m}{r_1, r_2, \ldots, r_t} = \sum_{k_1, k_2, \ldots, k_t} \binom{n}{k_1, k_2, \ldots, k_t} \binom{m}{r_1 - k_1, r_2 - k_2, \ldots, r_t - k_t}. \tag{4}$$

Proof. For a combinatorial proof see Theorem 2.

We sketch an algebraic proof, it suffices to develop

$$(x_1 + x_2 + \cdots + x_t)^{n+m} = (x_1 + x_2 + \cdots + x_t)^n (x_1 + x_2 + \cdots + x_t)^m$$

and to identify the coefficient of $x_1^{r_1} x_2^{r_2} \cdots x_t^{r_t}$ for both sides. ■

Now, give the generalized multinomial Chu-Vandermonde identity.

Theorem 2 *Let $t, s \in \mathbb{N}$ and $r_1, r_2, \ldots, r_t, n_1, n_2, \ldots, n_s$ be nonnegative integers, the following identity holds*

$$\binom{n_1 + n_2 + \cdots + n_s}{r_1, r_2, \ldots, r_t} = \sum_{k_{ij}} \binom{n_1}{k_{11}, k_{12}, \ldots, k_{1t}} \cdots \binom{n_s}{k_{s1}, k_{s2}, \ldots, k_{st}} \tag{5}$$

where the summation is taken over all $k_{ij},\ i = 1, \ldots, s;\ j = 1, \ldots, t$ such that $k_{1l} + k_{2l} + \cdots + k_{sl} = r_l,\ l = 1, \ldots, t$.

Proof. We sketch an algebraic proof, it suffices to develop

$$(x_1 + x_2 + \cdots + x_t)^{n_1 + \cdots + n_s} = (x_1 + x_2 + \cdots + x_t)^{n_1} \cdots (x_1 + x_2 + \cdots + x_t)^{n_s}$$

and to identify the coefficient of $x_1^{r_1} x_2^{r_2} \cdots x_t^{r_t}$ for both sides.

For a combinatorial proof: consider s different nationalities of students in the university with t levels of learning: the first year, the second year, ..., and the t^{th} year. From students composed by n_1 of nationality 1,...,n_s of nationality s ; we want to choose r_1 students of the first year, r_2 students of the second year,..., and r_t students of the t^{th} year. We do it by summing over all possible values of $k_{i,1}$ of the 1^{st} year ; $k_{i,2}$ of the 2^{nd} year; ... ; $k_{i,t}$ of the t^{th} year of nationality i for $i = 1, \ldots, s$. such that the sum of student of the same year j correspond to r_j. ■

3 A poly-multi-hypergeometric distribution probability

Where both sides of (5) are divided by the expression on the left, the sum be 1, in this case also we can interpret the terms of the sum as probabilities. The resulting probability distribution can be "named" as "*poly-multi-hypergeometric distribution*". "poly" for $n_1, n_2, \ldots, n_s$ and "multi" for $r_1, r_2, \ldots, r_t$.

Here we formulate an other example, to express the probability distribution that is the probability distribution of r_1 balls with number 1, r_2 balls with number 2, …, and r_t balls with number t from an urn containing N balls with proportion p_1 for the color 1, p_2 for the color 2, …, p_s for the color s,

$$K = \begin{bmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,t} \\ k_{2,1} & k_{2,2} & & k_{2,t} \\ \vdots & \vdots & & \vdots \\ k_{s,1} & k_{s,2} & \cdots & k_{s,t} \end{bmatrix} \begin{matrix} n_1 = Np_1 \\ n_2 = Np_2 \\ \\ n_s = Np_s \end{matrix}$$

then the probability to get a matrix distribution as a contingence table brewing the lines sums and the column sums is given by

$$P(K) = \frac{\binom{Np_1}{k_{1,1}, k_{1,2}, \ldots, k_{1,t}} \cdots \binom{Np_s}{k_{s,1}, k_{s,2}, \ldots, k_{s,t}}}{\binom{N}{r_1, r_2, \ldots, r_s}}, \tag{6}$$

We can consider this example as a way to introduce a random matrix. Also, by normalizing, we notice that K can be viewed as a double stochastic matrix.

4 Complex variant of Chu-Vandermonde identity

The identity (4) generalizes to non-integer arguments. We have to specify the way of this extension. Let $x \in \mathbb{C}$ and $r \in \mathbb{Z}$,we define

$$\binom{x}{r} = \begin{cases} \frac{1}{r!} x(x-1)\cdots(x-r+1), & r \geq 1 \\ 1, & r = 0 \\ 0, & r < 0 \end{cases}.$$

For $r > 0$, it is a polynomial of degree r.

$$\begin{aligned} & \binom{x}{r_1, r_2, \ldots, r_{t-1}, x - \sum_j r_j} \\ = & \binom{x}{r_1}\binom{x-r_1}{r_2}\cdots\binom{x-r_1-\cdots-r_{t-1}}{r_{t-1}}, \\ = & \frac{x(x-1)\cdots(x-r_1+1)}{r_1!}\frac{(x-r_1)\cdots(x-r_1-r_2+1)}{r_2!}\cdots \\ & \qquad \cdots\frac{(x-r_1-\cdots-r_{t-2})\cdots(x-r_1-\cdots-r_{t-1}+1)}{r_{t-1}!}, \end{aligned}$$

it is a polynomial of degree $r_1 + r_2 + \cdots + r_{t-1}$.

Lemma 3 *We have*

$$\binom{x}{r_1, r_2, \ldots, r_{t-1}, x-r} = \binom{r}{r_1, \ldots, r_{t-1}} \binom{x}{r}. \tag{7}$$

It is implicit that $r = \sum_{j=1}^{t-1} r_j$.

It is well known that for general complex valued x and y, Chu-Vandermonde identity takes the following form

$$\binom{x+y}{r} = \sum_{k=0}^{\infty} \binom{x}{k} \binom{y}{r-k}$$

Theorem 4 *For all* x *and* y *complex numbers and all* $r_1, r_2, \ldots, r_{t-1}$ *nonnegative integers, we set* $r = r_1 + \cdots + r_{t-1}$ *we have the following*

$$\begin{aligned} &\binom{x+y}{r_1, \ldots, r_{t-1}, x+y-r} \\ = &\sum_{k_1, \ldots, k_{t-1}} \binom{x}{k_1, \ldots, k_{t-1}, x - \sum_j k_j} \binom{y}{r_1 - k_1, \ldots, r_{t-1} - k_{t-1}, y - r + \sum_j k_j} \end{aligned} \tag{8}$$

Proof. Set $\sum_j k_j = k$, by Lemma3 and Theorem1, we have

$$\begin{aligned} &\binom{x+y}{r_1, \ldots, r_{t-1}, x+y-r} \\ = &\binom{r}{r_1, \ldots, r_{t-1}} \binom{x+y}{r} \\ = &\binom{r}{r_1, \ldots, r_{t-1}} \sum_k \binom{x}{k} \binom{y}{r-k} \\ = &\sum_{k_1, \ldots, k_{t-1}} \binom{l}{k_1, \ldots, k_{t-1}} \binom{r-l}{r_1 - k_1, \ldots, r_{t-1} - k_{t-1}} \sum_k \binom{x}{k} \binom{y}{r-k} \end{aligned}$$

In particular, for $l = k$

$$\begin{aligned} &\binom{x+y}{r_1, \ldots, r_{t-1}, x+y-r} \\ = &\sum_{k \geq 0} \sum_{\substack{k_1, \ldots, k_{t-1} \\ \sum k_j = k}} \binom{k}{k_1, \ldots, k_{t-1}} \binom{x}{k} \binom{r-k}{r_1 - k_1, \ldots, r_{t-1} - k_{t-1}} \binom{y}{r-k}, \end{aligned}$$

we conclude by the Lemma. ■

Now, we give the complex version of the generalized multinomial Chu-Vandermonde identity.

Theorem 5 *For all* $x_1, \ldots, x_s$ *complex numbers and all* $r_1, r_2, \ldots, r_{t-1}$ *nonnegative integers, set* $r = \sum_{j=1}^{t-1} r_j$ *and* $x = \sum_{j=1}^{s} x_j$*, we have the following*

$$\begin{aligned} &\binom{x_1 + \cdots + x_s}{r_1, \ldots, r_{t-1}, x - r} \\ = &\sum_{k_{i,j}} \binom{x_1}{k_{1,1}, \ldots, k_{1,t-1}, x_1 - \sum_j k_{1,j}} \cdots \binom{x_s}{k_{s,1}, \ldots, k_{s,t-1}, x_s - \sum_j k_{s,j}}, \end{aligned}$$

where the summation is taken aver all $k_{i,j}, i = 1, \ldots, s,\ j = 1, \ldots, t-1$ *such that* $k_{1,l} + k_{2,l} + \cdots + k_{s,l} = r_l,\ l = 1, \ldots, t-1.$

Proof. We leave the proof to the reader, it suffices to consider the proof of Theorem4 for s arguments. ■

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