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The three different forms of the periods of the Morgan-Voyce sequence modulo odd primes

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Abstract: In this paper we give the three different forms of the periods of the Morgan-Voyce sequence modulo odd primes p.

Keywords: Periods of sequences, Morgan-Voyce sequence, prime numbers, congruences.

Résumé : Dans cet article, nous donnons les trois formes différentes des périodes de la suite de Morgan-Voyce modulo les nombres premiers impairs p.

Mots clés : Périodes de suites, suites de Morgan-Voyce, nombres premiers, congruences.

1 Introduction

In dealing with electrical ladder networks, Morgan-Voyce [7] defined the two sequences $(V_n)_n$ and $(M_n)_n$ by

$$\begin{cases} V_0 = 1,\ V_1 = 1, \\ V_n = (2+t)V_{n-1} - V_{n-2}, \qquad (n \geq 2), \end{cases}$$

and

$$\begin{cases} M_0 = 0,\ M_1 = 1, \\ M_n = (2+t)M_{n-1} - M_{n-2}, \qquad (n \geq 2), \end{cases} \tag{1}$$

where t is a parameter that we assume to be in $\mathbb{Z}$. As for the Fibonacci sequence, each element of the Morgan-Voyce sequences $(V_n)_{n\geq1}$ and $(M_n)_{n\geq1}$ can be expressed in Pascal's triangle as follows: [1, 2, 3, 8]

$$V_n = \sum_{k=0}^{n-1} \binom{n+k-1}{2k} t^k, \qquad (n \geq 1),$$

and

$$M_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} t^k, \qquad (n \geq 1). \tag{2}$$

The coefficients $\binom{n+k-1}{2k}$ and $\binom{n+k}{2k+1}$ are well known in the study of electrical networks. The coefficients $\binom{n+k-1}{2k}$ are exactly the lines of the DFF triangle given in Table 1 below [4], while The coefficients $\binom{n+k}{2k+1}$ form the lines of the DFFz triangle given in Table 2 below [5].

$n\backslash k$	0	1	2	3	4	5	$\dots$
1	1						
2	1	1					
3	1	3	1				
4	1	6	5	1			
5	1	10	15	7	1		
6	1	15	35	28	9	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1. DFF triangle.

$n\backslash k$	0	1	2	3	4	5	$\dots$
1	1						
2	2	1					
3	3	4	1				
4	4	10	6	1			
5	5	20	21	8	1		
6	6	35	56	36	10	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 2. DFFz triangle.

In this paper, we look at the form of the Morgan-Voyce sequence $(M_n)_n$ modulo an odd prime p. The companion matrix of this sequence is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 2+t \end{smallmatrix}\right)$. This matrix is of determinant 1 and is therefore invertible modulo any integer $m \geq 2$. We deduce that the Morgan-Voyce sequence modulo m is simply periodic [9]. We denote the period of the sequence $(M_n \bmod m)_{n\geq0}$ by $k(m)$, i.e., the smallest positive integer k such that $M_k \equiv 0 \bmod m$ and $M_{k+1} \equiv 1 \bmod m$. We denote by $d(m)$ the smallest positive integer k such that $M_k \equiv 0 \bmod m$.

The periodic properties of the Morgan-Voyce sequence have been studied in [1]. We recall that the terms for which $M_n \equiv 0 \bmod m$ have subscripts that form an simple arithmetic progression [1, Theorem 2.4]. Thus we have

$$M_n \equiv 0 \bmod m \iff d(m) \mid n, \tag{3}$$

we deduce that $d(m)$ divides $k(m)$. We define a function $l(m)$ by the equation $d(m)l(m) = k(m)$, note that $l(m)$ is an integer for all $m \geq 2$, it is the number of zeros in a period of the sequence $(M_n \bmod m)_{n\geq 0}$.

We also recall the following:

Lemma 1 *$l(m)$ is the exponent to which $(-M_{d(m)-1})$ belongs modulo m, i.e., $l(m)$ is the smallest positive integer n for which $(-1)^n M^n_{d(m)-1} \equiv 1 \mod m$.*

Proof. See [1, Lemma 2.3] ■

Theorem 2 *Let $m > 2$ be an integer and $p \neq 2$ a prime.*

(a) $l(2) = 1$ and $l(m) = 1$ or 2.

(b) $l(p) = 2$ if and only if $k(p)$ is even, in this case $d(p) = k(p)/2$.

(c) $l(p) = 1$ if and only if $k(p)$ is odd, in this case $d(p) = k(p) \neq p \pm 1$.

(d) If m has prime factorization $\prod p_i^{e_i}$, then $d(m) = \operatorname{lcm}\big(d(p_i^{e_i})\big)$.

Proof. See [1, Theorem 2.5] ■

Let $F(x) = x^2 - (2+t)x + 1$ be the characteristic polynomial of the sequence $(M_n)_n$ and let $\Delta = t^2 + 4t$ its discriminant. The complex roots of $F(x)$ are $\alpha = 1/2\Big((2+t) + \sqrt{\Delta}\Big)$ and $\beta = 1/2\Big((2+t) - \sqrt{\Delta}\Big)$. If $\Delta \neq 0$ (i.e., $t \neq 0, -4$), we have Binet's formula [6]

$$M_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad (n \geq 0). \tag{4}$$

Identity (4) gives a natural extension of the sequence $(M_n)_{n\geq 0}$ to negative values of n. Using the identity $\alpha^n \beta^n = 1$, we find

$$M_{-n} = -M_n. \tag{5}$$

If $t = 0$, then $M_n = n$ for any $n \geq 0$ and if $t = -4$, then $M_n = (-1)^{n+1} n$ for any $n \geq 0$. It is easy to see that Identity (5) holds also for $t \in \{0, -4\}$. We deduce that the recurrence relation $M_n = (2+t)M_{n-1} - M_{n-2}$ holds for the extended sequence for any $t \in \mathbb{Z}$.

Let $m \geq 2$ be an integer. Since we have Identity (5), one can show by induction on n that

$$M_{k(m)-n} \equiv -M_n \bmod m, \qquad (n \geq 0). \tag{6}$$

We deduce from Identity (6) the following proposition which tells us that the sequence $(M_n \bmod p)_n$, where p is an odd prime, has one of three forms given in the three tables below which are given for $t = 1$.

Proposition 3 *Let p be an odd prime.*

1. *If $l(p) = 1$, then*
$$M_{k(p)-i} \equiv -M_i \mod p, \qquad 0 \leq i \leq (k(p)-1)/2. \tag{7}$$

2. *If $l(p) = 2$, then*
$$M_{d(p)-i} \equiv M_i \mod p, \qquad 0 \leq i \leq \lfloor d(p)/2 \rfloor, \tag{8}$$
and
$$M_{k(p)-i} \equiv -M_i \mod p, \qquad 0 \leq i \leq d(p)-1. \tag{9}$$

Proof. Assertion (1) follows from the Identity (6). To show assertion (2), assume that $l(p) = 2$, then $d(p) = k(p)/2$. We get from Lemma 1 that $M^2_{d(p)-1} \equiv 1 \bmod p$, which is equivalent to $M_{d(p)-1} \equiv \pm 1 \bmod p$. Assume that $M_{d(p)-1} \equiv -1 \bmod p$, then $M_{d(p)+1} = (2+t)M_{d(p)} - M_{d(p)-1} \equiv 1 \bmod p$, hence $k(p)|d(p)$ which is a contradiction. Thus, we must have $M_{d(p)-1} \equiv M_1 \bmod p$. Since $M_{d(p)} \equiv M_0 \bmod p$, one can show by induction on $i \geq 0$ that
$$M_{d(p)-i} \equiv M_i \bmod p, \qquad i \geq 0,$$
from which we deduce (8). Identity (9) follows from Identity (6). ■

Remark 1 *In case (1) of Proposition 3, the sequence $(M_n \bmod p)_n$ is of the form*
$$M_0, M_1, M_2, \ldots, M_{(k(p)-1)/2}, -M_{(k(p)-1)/2}, \ldots, -M_2, -M_1; M_0, M_1, \ldots,$$
which corresponds to the examples in Table 1. In case (2) of Proposition 3, a period is composed of two opposite sign palindromes and are given as follows: if $d(p)$ is even, the sequence $(M_n \bmod p)_n$ is of the form
$$M_0, M_1, \ldots, M_{d(p)/2-1}, M_{d(p)/2}, M_{d(p)/2-1}, \ldots, M_{d(p)-1} \equiv M_1, M_{d(p)} \equiv M_0,$$
$$-M_1, \ldots, -M_{d(p)/2-1}, -M_{d(p)/2}, -M_{d(p)/2-1}, \ldots, -M_1; M_0, M_1, \ldots,$$
which corresponds to the examples in Table 2. If $d(p)$ is odd, the sequence $(M_n \bmod p)_n$ is of the form
$$M_0, M_1, \ldots, M_{(d(p)-1)/2}, M_{(d(p)-1)/2}, \ldots, M_{d(p)-1} \equiv M_1, M_{d(p)} \equiv M_0,$$
$$-M_1, \ldots, -M_{(d(p)-1)2}, -M_{(d(p)-1)/2}, \ldots, -M_1; M_0, M_1, \ldots,$$
which corresponds to the examples in Table 3.

p	$(M_n \bmod p)_n$	$k(p)$
11	$0, 1, 3, 8, 10; 0, 1$	5
19	$0, 1, 3, 8, 2, 17, 11, 16, 18; 0, 1$	9
29	$0, 1, 3, 8, 21, 26, 28; 0, 1$	7
31	$0, 1, 3, 8, 21, 24, 20, 5, 26, 11, 7, 10, 23, 28, 30; 0, 1$	15
53	$0, 1, 3, 8, 21, 2, 38, 6, 33, 40, 34, 9, 46, 23, 30, 7, 44, 19, 13, 20, 47, 15, 51, 32, 45, 50, 52; 0, 1$	27
59	$0, 1, 3, 8, 21, 55, 26, 23, 43, 47, 39, 11, 53, 30, 37, 22, 29, 6, 48, 20, 12, 16, 36, 33, 4, 38, 51, 56, 58; 0, 1$	29
79	$0, 1, 3, 8, 21, 55, 65, 61, 39, 56, 50, 15, 74, 49, 73, 12, 42, 35, 63, 75, 4, 16, 44, 37, 67, 6, 30, 5, 64, 29, 23, 40, 18, 14, 24, 58, 71, 76, 78; 0, 1$	39
101	$0, 1, 3, 8, 21, 55, 43, 74, 78, 59, 99, 36, 9, 92, 65, 2, 42, 23, 27, 58, 46, 80, 93, 98, 100; 0, 1$	25
131	$0, 1, 3, 8, 21, 55, 13, 115, 70, 95, 84, 26, 125, 87, 5, 59, 41, 64, 20, 127, 99, 39, 18, 15, 27, 66, 40, 54, 122, 50, 28, 34, 74, 57, 97, 103, 81, 9, 77, 91, 65, 104, 116, 113, 92, 32, 4, 111, 67, 90, 72, 126, 44, 6, 105, 47, 36, 61, 16, 118, 76, 110, 123, 128, 130; 0, 1$	65
139	$0, 1, 3, 8, 21, 55, 5, 99, 14, 82, 93, 58, 81, 46, 57, 125, 40, 134, 84, 118, 131, 136, 138; 0, 1$	23
151	$0, 1, 3, 8, 21, 55, 144, 75, 81, 17, 121, 44, 11, 140, 107, 30, 134, 70, 76, 7, 96, 130, 143, 148, 150; 0, 1$	25
181	$0, 1, 3, 8, 21, 55, 144, 15, 82, 50, 68, 154, 32, 123, 156, 164, 155, 120, 24, 133, 13, 87, 67, 114, 94, 168, 48, 157, 61, 26, 17, 25, 58, 149, 27, 113, 131, 99, 166, 37, 126, 160, 173, 178, 180; 0, 1$	45
191	$0, 1, 3, 8, 21, 55, 144, 186, 32, 101, 80, 139, 146, 108, 178, 44, 145, 9, 73, 19, 175, 124, 6, 85, 58, 89, 18, 156, 68, 48, 76, 180, 82, 66, 116, 91, 157, 189, 28, 86, 39, 31, 54, 131, 148, 122, 27, 150, 41, 164, 69, 43, 60, 137, 160, 152, 105, 163, 2, 34, 100, 75, 125, 109, 11, 115, 143, 123, 35, 173, 102, 133, 106, 185, 67, 16, 172, 118, 182, 46, 147, 13, 83, 45, 52, 111, 90, 159, 5, 47, 136, 170, 183, 188, 190; 0, 1$	95

Table 1: Periods modulo small prims p for which $k(p)$ is odd.

p	$(M_n \bmod p)_n$	$k(p)$
3	$0, 1, 0, 2; 0, 1$	4
7	$0, 1, 3, 1, 0, 6, 4, 6; 0, 1$	8
23	$0, 1, 3, 8, 21, 9, 6, 9, 21, 8, 3, 1, 0, 22, 20, 15, 2, 14, 17, 14, 2, 15, 20, 22; 0, 1$	24
41	$0, 1, 3, 8, 21, 14, 21, 8, 3, 1, 0, 40, 38, 33, 20, 27, 20, 33, 38, 40; 0, 1$	20
43	$0, 1, 3, 8, 21, 12, 15, 33, 41, 4, 14, 38, 14, 4, 41, 33, 15, 12, 21, 8, 3, 1, 0, 42, 40, 35, 22, 31, 28, 10, 2, 39, 29, 5, 29, 39, 2, 10, 28, 31, 22, 35, 40, 42; 0, 1$	44
47	$0, 1, 3, 8, 21, 8, 3, 1, 0, 46, 44, 39, 26, 39, 44, 46; 0, 1$	16
83	$0, 1, 3, 8, 21, 55, 61, 45, 74, 11, 42, 32, 54, 47, 4, 48, 57, 40, 63, 66, 52, 7, 52, 66, 63, 40, 57, 48, 4, 47, 54, 32, 42, 11, 74, 45, 61, 55, 21, 8, 3, 1, 0, 82, 80, 75, 62, 28, 22, 38, 9, 72, 41, 51, 29, 36, 79, 35, 26, 43, 20, 17, 31, 76, 31, 17, 20, 43, 26, 35, 79, 36, 29, 51, 41, 72, 9, 38, 22, 28, 62, 75, 80, 82; 0, 1$	84
103	$0, 1, 3, 8, 21, 55, 41, 68, 60, 9, 70, 98, 18, 59, 56, 6, 65, 86, 90, 81, 50, 69, 54, 93, 19, 67, 79, 67, 19, 93, 54, 69, 50, 81, 90, 86, 65, 6, 56, 59, 18, 98, 70, 9, 60, 68, 41, 55, 21, 8, 3, 1, 0, 102, 100, 95, 82, 48, 62, 35, 43, 94, 33, 5, 85, 44, 47, 97, 38, 17, 13, 22, 53, 34, 49, 10, 84, 36, 24, 36, 84, 10, 49, 34, 53, 22, 13, 17, 38, 97, 47, 44, 85, 5, 33, 94, 43, 35, 62, 48, 82, 95, 100, 102; 0, 1$	104
107	$0, 1, 3, 8, 21, 55, 37, 56, 24, 16, 24, 56, 37, 55, 21, 8, 3, 1, 0, 106, 104, 99, 86, 52, 70, 51, 83, 91, 83, 51, 70, 52, 86, 99, 104, 106; 0, 1$	36

Table 2: Periods modulo small prims p for which $k(p)$ and $d(p)$ are even.

p	$(M_n \bmod p)_n$	$k(p)$
13	$0,1,3,8,8,3,1,0,12,10,5,5,10,12;0,1$	14
17	$0,1,3,8,4,4,8,3,1,0,16,14,9,13,13,9,14,16;0,1$	18
37	$0,1,3,8,21,18,33,7,25,31,31,25,7,33,18,21,8,3,1,0,36,34,29,16,19,4,30,12,$ $6,6,12,30,4,19,16,29,34,36;0,1$	38
61	$0,1,3,8,21,55,22,11,11,22,55,21,8,3,1,0,60,58,53,40,6,39,50,50,39,6,40,53,$ $58,60;0,1$	30
73	$0,1,3,8,21,55,71,12,38,29,49,45,13,67,42,59,62,54,27,27,54,62,59,42,67,13,$ $45,49,29,38,12,71,55,21,8,3,1,0,72,70,65,52,18,2,61,35,44,24,28,60,6,31,$ $14,11,19,46,46,19,11,14,31,6,60,28,24,44,35,61,2,18,52,65,70,72;0,1$	74
89	$0,1,3,8,21,55,55,21,8,3,1,0,88,86,81,68,34,34,68,81,86,88;0,1$	22
97	$0,1,3,8,21,55,47,86,17,62,72,57,2,46,39,71,77,63,15,79,28,5,84,53,75,75,53,$ $84,5,28,79,15,63,77,71,39,46,2,57,72,62,17,86,47,55,21,8,3,1,0,96,94,89,76,$ $42,50,11,80,35,25,40,95,51,58,26,20,34,82,18,69,92,13,44,22,22,44,13,92,$ $69,18,82,34,20,26,58,51,95,40,25,35,80,11,50,42,76,89,94,96;0,1$	98
109	$0,1,3,8,21,55,35,50,6,77,7,53,43,76,76,43,53,7,77,6,50,35,55,21,8,$ $3,1,0,108,106,101,88,54,74,59,103,32,102,56,66,33,33,66,56,$ $102,32,103,59,74,54,88,101,106,108;0,1$	54
113	$0,1,3,8,21,55,31,38,83,98,98,83,38,31,55,21,8,3,1,0,112,110,105,92,58,82,$ $75,30,15,15,30,75,82,58,92,105,110,112;0,1$	38
149	$0,1,3,8,21,55,144,79,93,51,60,129,29,107,143,24,78,61,105,105,61,78,24,$ $143,107,29,129,60,51,93,79,144,55,21,8,3,1,0,148,146,141,128,94,5,$ $70,56,98,89,20,120,42,6,125,71,88,44,44,88,71,125,6,42,120$ $20,89,98,56,70,5,94,128,141,146,148;0,1$	74

Table 3: Periods modulo small prims p for which $k(p)$ is even and $d(p)$ is odd.

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