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Some applications of the generalized Bell umbra in congruences

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Abstract: In this paper we give some congruences on the r-derangement polynomials (defined below), Lah polynomials and some versions of Bell numbers and polynomials.

Keywords: Generalized Bell umbra, Bell polynomials, derangement polynomials, Lah polynomials, congruences.

Résumé : Dans ce papier, on donne quelques congruences sur les polynômes r-derangement (définés ci-dessous), les polynômes de Lah et quelques versions des nombres et polynômes de Bell.

Mots clés : L'ombra généralisé de Bell, polynômes de Bell, polynômes de derangement, polynômes de Lah, congruences.

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1 Introduction

Recall that the unsigned r-Stirling numbers $\left[{n \atop k}\right]_r$ of the first kind and the r-Stirling numbers $\left\{{n \atop k}\right\}_r$ of the the second kind [1], are defined by

$$(x)_n = \sum_{k=0}^{n} (-1)^{n-k} \left[{n+r \atop k+r}\right]_r (x+r)^k, \quad (x+r)^n = \sum_{k=0}^{n} \left\{{n+r \atop k+r}\right\}_r (x)_k,$$

where $(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$ if $n \geq 1$ and $(\alpha)_0 = 1$.
Let $\mathbf{B}_\mathbf{x}^n = \mathcal{B}_n(x)$ be the generalized Bell umbra introduced by Sun et al. [14]. This leads that the r-Bell polynomials $\mathcal{B}_{n,r}(x)$ can be written as

$$\mathcal{B}_{n,r}(x) = \sum_{k=0}^{n} \left\{{n+r \atop k+r}\right\}_r x^k = \sum_{j=0}^{n} \binom{n}{j} r^j \mathcal{B}_{n-j}(x) = (\mathbf{B}_\mathbf{x} + r)^n. \quad (1)$$

Congruences on Bell and derangement numbers and polynomials have been studied by several researchers [4, 5, 14, 15]. We are interested in such studies using umbral calculus [5, 11, 12]. In this paper, we investigate the properties of the generalized Bell umbra to derive new congruences involving the derangement polynomials, Lah polynomials and some versions of the Bell numbers and polynomials. This work is motivated by the application of the Bell umbra and its generalization in congruences studied in [5, 14, 15].
The paper is organized as follow: In the next section we give some properties of the generalized Bell umbra. In the third section we introduce the r-derangement polynomials by showing their connections to Bell polynomials and we give some congruences on these polynomials. In the last three sections we present some congruences on the Lah polynomials and on some versions of the Bell numbers and Bell polynomials and their connections to the derangement polynomials. In the remainder of this paper, for any polynomials f and g, we denote by $f(x) \equiv g(x)$ to mean $f(x) \equiv g(x) \pmod{p\mathbb{Z}_p[x]}$ and for any numbers a and b by $a \equiv b$ we mean $a \equiv b \pmod{p}$.

2 Some properties of the generalized Bell umbra

It is known [14] that for any polynomial f and integer $n \geq 0$, there hold

$$\mathbf{B}_\mathbf{x}^{n+1} = x(\mathbf{B}_\mathbf{x} + 1)^n \quad \text{and} \quad (\mathbf{B}_\mathbf{x})_n f(\mathbf{B}_\mathbf{x}) = x^n f(\mathbf{B}_\mathbf{x} + n). \quad (2)$$

In particular, for $f(x) = 1$ in the last identity we get

$$(\mathbf{B}_\mathbf{x})_n = x^n. \quad (3)$$

It is known that the n-th Bell polynomial admits the stated relation by means of Dobinski's formula $\mathcal{B}_n(x) = e^{-x} \sum_{j\geq 0} j^n \frac{x^j}{j!}$ which leads to

Lemma 1 *For any polynomial f, there holds*

$$f(\mathbf{B}_\mathbf{x}) = e^{-x} \sum_{j\geq 0} f(j) \frac{x^j}{j!} \quad (4)$$

For example, when we replace $f(x)$ in Lemma 1 by $(x)_n f(x)$, we obtain the second identity of (2). Also, if $(P_n(x))$ be a sequence of polynomials, then by taking $f(x) = P_{n+1}(x) - 2P_n(x) + P_{n-1}(x)$ in Lemma (1) we obtain:

Corollary 2 *Let $(P_n(x))$ be a sequence of polynomials. If $(P_n(x))$ is concave (resp. convex) then $(P_n(\mathbf{B_x}))_{n\geq 0}$ is concave (resp. convex).*

Proposition 3 *Let $n \geq 0, m \geq 0, s \geq 1$ be integers, let p be a prime number and let f be a polynomial in $\mathbb{Z}[x]$. Then*

$$f(\mathbf{B_x})\left(\mathbf{B_x^{p^s}} - \mathbf{B_x}\right) \equiv \left(x^p + x^{p^2} + \cdots + x^{p^s}\right) f(\mathbf{B_x}).$$

Proof. It suffices to take $f(x) = x^n$ and proceed by induction on s. Indeed, for $s = 1$, use the known congruence $B_{n+p}(x) \equiv B_{n+1}(x) + x^p B_n(x)$ to get $\mathbf{B_x^n}(\mathbf{B_x^p} - \mathbf{B_x}) = \mathbf{B_x^{n+p}} - \mathbf{B_x^{n+1}} = B_{n+p}(x) - B_{n+1}(x) \equiv x^p B_n(x) = x^p \mathbf{B_x^n}$. Assume it is true for s. Then

$$\begin{aligned}
\mathbf{B_x^n}\left(\mathbf{B_x^{p^{s+1}}} - \mathbf{B_x}\right) &= \left(\left(\mathbf{B_x^{p^s}} - \mathbf{B_x} + \mathbf{B_x}\right)^p - \mathbf{B_x}\right)\mathbf{B_x^n} \\
&\equiv \left(\left(\mathbf{B_x^{p^s}} - \mathbf{B_x}\right)^p + \mathbf{B_x^p} - \mathbf{B_x}\right)\mathbf{B_x^n} \\
&= \left(\mathbf{B_x^{p^s}} - \mathbf{B_x}\right)^p \mathbf{B_x^n} + \left(\mathbf{B_x^p} - \mathbf{B_x}\right)\mathbf{B_x^n} \\
&\equiv \left(x^p + x^{p^2} + \cdots + x^{p^s}\right)\left(\mathbf{B_x^{p^s}} - \mathbf{B_x}\right)^{p-1}\mathbf{B_x^n} + x^p \mathbf{B_x^n} \\
&\equiv \cdots \\
&\equiv \left(x^p + x^{p^2} + \cdots + x^{p^s}\right)^p \mathbf{B_x^n} + x^p \mathbf{B_x^n} \\
&\equiv \left(x^{p^2} + x^{p^3} + \cdots + x^{p^{s+1}}\right)\mathbf{B_x^n} + x^p \mathbf{B_x^n} \\
&= \left(x^p + x^{p^2} + \cdots + x^{p^{s+1}}\right)\mathbf{B_x^n},
\end{aligned}$$

hence, the proof is complete. ■

Corollary 4 *Let $n \geq 0, m \geq 0, s \geq 1$ be integers, let p be a prime number and let f be a polynomial in $\mathbb{Z}[x]$. Then*

$$\begin{aligned}
f(\mathbf{B_x})\left(\mathbf{B_x^p} - \mathbf{B_x}\right) &\equiv x^p f(\mathbf{B_x}), \\
\mathbf{B_x^p} - \mathbf{B_x} &\equiv x^p, \\
B_{n+p^s}(x) &\equiv \left(x^p + x^{p^2} + \cdots + x^{p^s}\right) B_n(x) + B_{n+1}(x), \\
(\mathbf{B_x})_n\left(\mathbf{B_x^{p^s}} - \mathbf{B_x}\right) &\equiv \left(x^p + x^{p^2} + \cdots + x^{p^s}\right) x^n, \\
B_{mp^s+n}(x) &\equiv \sum_{k=0}^{m}\binom{m}{k}\left(x^p + x^{p^2} + \cdots + x^{p^s}\right)^k B_{n+m-k}(x).
\end{aligned}$$

Proof. Take $s = 1$, $f(x) = 1, x^n$ or $(x)_n$ in Proposition 3 and the last congruence follows from $B_{mp^s+n}(x) = \mathbf{B_x^n}\left(\mathbf{B_x^{p^s}} - \mathbf{B_x} + \mathbf{B_x}\right)^m = \sum_{k=0}^{m}\binom{m}{k}\mathbf{B_x^{n+m-k}}\left(\mathbf{B_x^{p^s}} - \mathbf{B_x}\right)^k$. ■

3 Congruences on the r-derangement polynomials

Let $\left[{n \atop j}\right]_r^{2\uparrow}$ be the number of permutations of the set $[n] := \{1, \ldots, n\}$ into j cycles without fixed points such that the elements of the set $[r]$ are in different cycles and set $\left[{n \atop j}\right]^{2\uparrow} = \left[{n \atop j}\right]_0^{2\uparrow}$. The number of derangements of the set $[n]$ is $\mathcal{D}_n = \sum_{j=0}^{n} \left[{n \atop j}\right]^{2\uparrow}$ and we define the r-derangement number by $\mathcal{D}_{n,r} = \sum_{j=0}^{n} \left[{n+r \atop j+r}\right]_r^{2\uparrow}$. The derangement polynomial $\mathcal{D}_n(x)$ [3, 13] can be extended to the r-derangement polynomial defined by

$$\mathcal{D}_{n,r}(x) = \frac{1}{r!} \sum_{j=0}^{n} \binom{n}{j} (j+r)! \, (x-1)^{n-j}$$

for which we obtain $\mathcal{D}_n(x) = \mathcal{D}_{n,0}(x)$ and since [10]

$$\sum_{n \geq 0} \mathcal{D}_{n,r} \frac{t^n}{n!} = (1-t)^{-r-1} \exp(-t) = \sum_{n \geq 0} \mathcal{D}_{n,r}(0) \frac{t^n}{n!}$$

it follows that $\mathcal{D}_{n,r} = \mathcal{D}_{n,r}(0)$ which counts permutations of the set $[n+r]$ without fixed points and such that the elements of the set $[r]$ are in different cycles, hence $\mathcal{D}_n = \mathcal{D}_{n,0}$. Since the sequence $((x)_n)$ is of binomial type [2, 8], then by (3) we obtain

$$(\mathbf{B_x} - r)_n = \sum_{j=0}^{n} \binom{n}{j} (-r)_j (\mathbf{B_x})_{n-j} = (-1)^n \mathcal{D}_{n,r-1}(1-x). \tag{5}$$

So, the derangement polynomial $\mathcal{D}_n(x) = \mathcal{D}_{n,0}(x)$ satisfies

$$(\mathbf{B_x} - 1)_n = (-1)^n \mathcal{D}_n(1-x). \tag{6}$$

Now, we give some general congruences on the derangement polynomials and we start by giving the following:

Proposition 5 *For any prime number p and any integers $m, q \geq 0$, there holds*

$$\mathcal{D}_{mp+q,r-1}(1-x) \equiv (-x)^{mp} \mathcal{D}_{q,r-1}(1-x).$$

In particular, for $r = 1$ we get $\mathcal{D}_{mp+q}(1-x) \equiv (-x)^{mp} \mathcal{D}_q(1-x)$.

Proof. By Identity (5) and the congruence $\binom{p}{j} \equiv 0$, $1 \leq k \leq p-1$, we obtain

$$\begin{aligned}
\mathcal{D}_{p+q,r-1}(1-x) &= (-1)^{p+q} (\mathbf{B_x} - r)_{p+q} \\
&= (-1)^{p+q} (\mathbf{B_x} - r)_p (\mathbf{B_x} - r - p)_q \\
&= (-1)^{p+q} \sum_{j=0}^{p} \binom{p}{j} (-r)_{p-j} (\mathbf{B_x})_j (\mathbf{B_x} - r - p)_q \\
&\equiv (-1)^{p+q} (-r)_p (\mathbf{B_x} - r)_q + (-1)^{p+q} (\mathbf{B_x})_p (\mathbf{B_x} - r)_q \\
&\equiv (-1)^{p+q} x^p (\mathbf{B_x} + p - r)_q \\
&\equiv (-1)^{p+q} x^p (\mathbf{B_x} - r)_q = (-x)^p \mathcal{D}_{q,r-1}(1-x)
\end{aligned}$$

and one can proceed easily by induction on $m \geq 0$ to complete the proof. ■

Corollary 6 *Let $n \geq 0, m \geq 0, r \geq 1, s \geq 1$ be integers and p be a prime number. Then*

$$x^r \left(B_{p^s-1,r}(x) - 1\right) \equiv (-1)^{r-1} \left(x^p + x^{p^2} + \cdots + x^{p^s}\right) \mathcal{D}_{r-1}(1-x).$$

Proof. By (6) we get $RHS = \left(x^p + x^{p^2} + \cdots + x^{p^s}\right)(\mathbf{B_x} - 1)_{r-1}$. So, Proposition 3 gives $RHS \equiv (\mathbf{B_x} - 1)_{r-1}\left(\mathbf{B}_\mathbf{x}^{p^s} - \mathbf{B_x}\right) = (\mathbf{B_x})_r \left(\mathbf{B}_\mathbf{x}^{p^s-1} - 1\right) = x^r\left((\mathbf{B_x} + r)^{p^s-1} - 1\right) = LHS.$ ■

The following theorem gives a curious congruence on r-derangement polynomials.

Theorem 7 *Let $n, m, r \in \mathbb{N}$ with $r \geq 1$ and p be a prime number. Then*

$$x^{m+1}\sum_{k=0}^{p-1}(m+n+r)_{p-1-k}\,\mathcal{D}_{n+k,r-1}(1-x) \equiv (-1)^m \frac{x^p}{(r-1)!}\sum_{j=0}^{n}\binom{n}{j} \times (n-j+r-1)!\mathcal{D}_{m+j}(1-x).$$

In particular, for $r = 1$ we get

$$x^{m+1}\sum_{k=0}^{p-1}(m+n+1)_{p-1-k}\,\mathcal{D}_{n+k}(1-x) \equiv (-1)^m n! x^p \sum_{j=0}^{n}\frac{\mathcal{D}_{m+j}(1-x)}{j!}.$$

Proof. By the congruence $\binom{p-1}{k} \equiv (-1)^{p-1-k}$ and (5) we get

$$\begin{aligned} LHS &\equiv (-1)^n x^{m+1}\sum_{k=0}^{p-1}\binom{p-1}{k}(m+n+r)_{p-1-k}(\mathbf{B_x} - r)_{n+k} \\ &= (-1)^n x^{m+1}\sum_{k=0}^{p-1}\binom{p-1}{k}(m+n+r)_{p-1-k}(\mathbf{B_x} - r - n)_k (\mathbf{B_x} - r)_n \\ &= (-1)^n x^{m+1}(\mathbf{B_x} + m)_{p-1}(\mathbf{B_x} - r)_n \end{aligned}$$

and by (2) and Proposition 3 the above congruence becomes

$$\begin{aligned} LHS &= (-1)^n (\mathbf{B_x})_{m+1}(\mathbf{B_x} - 1)_{p-1}(\mathbf{B_x} - r - m - 1)_n \\ &= (-1)^n (\mathbf{B_x} - 1)_m (\mathbf{B_x})_p (\mathbf{B_x} - r - m - 1)_n \\ &\equiv (-1)^n (\mathbf{B_x} - 1)_m (\mathbf{B}_\mathbf{x}^p - \mathbf{B_x})(\mathbf{B_x} - r - m - 1)_n \\ &\equiv (-1)^n x^p (\mathbf{B_x} - 1)_m (\mathbf{B_x} - r - m - 1)_n \\ &= (-1)^n x^p \sum_{j=0}^{n}\binom{n}{j}(-r)_{n-j}(\mathbf{B_x} - 1)_m (\mathbf{B_x} - m - 1)_j \\ &= (-1)^n x^p \sum_{j=0}^{n}\binom{n}{j}(-r)_{n-j}(\mathbf{B_x} - 1)_{m+j} = RHS, \end{aligned}$$

which complete the proof. ■

Corollary 8 *For any integer $n \in \mathbb{N}$ and any prime number p there hold*

$$\begin{aligned} x\sum_{k=0}^{p-1}\frac{\mathcal{D}_{n+k}(1-x)}{k!} &\equiv (-1)^n n!x^{n+2}\sum_{j=0}^{n}\frac{\mathcal{D}_{j+p-2-n}(1-x)}{j!},\ 0\leq n\leq p-2,\\ x\sum_{k=1}^{p-1}\frac{\mathcal{D}_{n+k}(1-x)}{(k-1)!} &\equiv (-1)^n n!x^{n+3}\sum_{j=0}^{n}\frac{\mathcal{D}_{j+p-3-n}(1-x)}{j!},\ 0\leq n\leq p-3.\end{aligned}$$

In particular, for $n=0$ or $n=1$ we obtain

$$\begin{aligned} x\sum_{k=0}^{p-1}\frac{\mathcal{D}_{k}(1-x)}{k!} &\equiv x^2\mathcal{D}_{p-2}(1-x),\quad x\sum_{k=1}^{p-1}\frac{\mathcal{D}_{k}(1-x)}{(k-1)!}\equiv x^3\mathcal{D}_{p-3}(1-x),\ p\geq 3,\\ x\sum_{k=0}^{p-1}\frac{\mathcal{D}_{k+1}(1-x)}{k!} &\equiv -x^3\left(\mathcal{D}_{p-3}(1-x)+\mathcal{D}_{p-2}(1-x)\right),\ p\geq 3,\\ x\sum_{k=1}^{p-1}\frac{\mathcal{D}_{k+1}(1-x)}{(k-1)!} &\equiv -x^4\left(\mathcal{D}_{p-4}(1-x)+\mathcal{D}_{p-3}(1-x)\right),\ p\geq 5.\end{aligned}$$

Proof. Take $r=1,\ m+n+1=p-1$ or $p-2$ in Theorem 7 and use the congruences: $k!\,(p-1)_{p-1-k}\equiv -1,\ 0\leq k\leq p-1$ and $(k-1)!\,(p-2)_{p-1-k}\equiv 1,\ 0\leq k\leq p-2.$ ■

4 Congruences on the Lah polynomials

In this section, we give some general congruences on the Lah polynomials. To start, let us give a short introduction to these polynomials. Recall that the (n,k)-th Lah number $L(n,k)$ counts partitions of the set $[n]$ into k ordered lists and the n-th Lah polynomial associated to the Lah numbers is defined by $\mathcal{L}_n(x)=\sum_{k=0}^{n}L(n,k)x^k$. From the known identity $\langle x\rangle_n=\sum_{k=0}^{n}L(n,k)\,(x)_k$, it results this polynomial can be written as

$$(\mathbf{B_x}+n-1)_n:=\langle\mathbf{B_x}\rangle_n=\sum_{k=0}^{n}L(n,k)\,(\mathbf{B_x})_k=\sum_{k=0}^{n}L(n,k)x^k=\mathcal{L}_n(x). \tag{7}$$

Proposition 9 *For any prime number p and any integers $m,q\geq 0$, there holds*

$$\mathcal{L}_{mp+q}(x)\equiv x^{mp}\mathcal{L}_q(x).$$

Proof. From Identity (7) we obtain

$$\begin{aligned}\mathcal{L}_{p+q}(x) &= \langle\mathbf{B_x}\rangle_{p+q}=\langle\mathbf{B_x}\rangle_p\langle\mathbf{B_x}+p\rangle_q\equiv\langle\mathbf{B_x}\rangle_p\langle\mathbf{B_x}\rangle_q\\ &= (\mathbf{B_x}+p-1)_p\langle\mathbf{B_x}\rangle_q\equiv(\mathbf{B_x}-1)_p\langle\mathbf{B_x}\rangle_q\\ &= (\mathbf{B_x}-p)(\mathbf{B_x}-1)_{p-1}\langle\mathbf{B_x}\rangle_q\equiv(\mathbf{B_x})_p\langle\mathbf{B_x}\rangle_q\\ &= x^p\langle\mathbf{B_x}+p\rangle_q\equiv x^p\langle\mathbf{B_x}\rangle_q=x^p\mathcal{L}_q(x).\end{aligned}$$

So, one can proceed easily by induction on $m\geq 0$ to complete the proof. ■

Theorem 10 *Let $n, m \in \mathbb{N}$ such that $m+n \geq 2$ and p be a prime number. Then*

$$(-x)^{m+n-2}\sum_{k=0}^{p-1}(-m)_{p-1-k}\,\mathcal{L}_{n+k}(x) \equiv x^{p-1}\sum_{j=0}^{n}(-1)^{j}\,L(n,j)\mathcal{D}_{j+m+n-2}(1-x)\,.$$

Proof. By the congruence $\binom{p-1}{k} \equiv (-1)^k$ and (5) we get

$$\begin{aligned}
LHS &= x^{m+n-2}\sum_{k=0}^{p-1}(-1)^k\langle m\rangle_{p-1-k}\,\mathcal{L}_{n+k}(x)\\
&\equiv x^{m+n-2}\sum_{k=0}^{p-1}\binom{p-1}{k}\langle m\rangle_{p-1-k}\langle\mathbf{B_x}\rangle_{n+k}\\
&= x^{m+n-2}\sum_{k=0}^{p-1}\binom{p-1}{k}\langle m\rangle_{p-1-k}\langle\mathbf{B_x}+n\rangle_{k}\langle\mathbf{B_x}\rangle_{n}\\
&= x^{m+n-2}\langle\mathbf{B_x}+m+n\rangle_{p-1}\langle\mathbf{B_x}\rangle_{n}\\
&= x^{m+n-2}(\mathbf{B_x}+m+n+p-2)_{p-1}(\mathbf{B_x}+n-1)_{n}
\end{aligned}$$

and, for any integer l, since

$$(\mathbf{B_x}+l+p)_{p-1} \equiv (\mathbf{B_x}+l)_{p-1}\ \texttt{and}\ (\mathbf{B_x})_n f(\mathbf{B_x}) = x^n f(\mathbf{B_x}+n)\,,\ f\in\mathbb{Z}[x]\,,$$

the above congruence becomes

$$\begin{aligned}
LHS &\equiv x^{m+n-2}(\mathbf{B_x}+m+n-2)_{p-1}(\mathbf{B_x}+n-1)_{n}\\
&= (\mathbf{B_x})_{p-1}(\mathbf{B_x})_{m+n-2}(\mathbf{B_x}-m+1)_{n}\\
&= x^{p-1}(\mathbf{B_x}+p-1)_{m+n-2}(\mathbf{B_x}-m+p)_{n}\\
&\equiv x^{p-1}(\mathbf{B_x}-1)_{m+n-2}(\mathbf{B_x}-m)_{n}\\
&\equiv x^{p-1}(\mathbf{B_x}-1)_{m+n-2}(\mathbf{B_x}-m-n+1+n-1)_{n}\\
&= x^{p-1}\sum_{j=0}^{n}\binom{n}{j}(n-1)_{n-j}(\mathbf{B_x}-1)_{m+n-2}(\mathbf{B_x}-m-n+1)_{j}\\
&= x^{p-1}\sum_{j=0}^{n}L(n,j)(\mathbf{B_x}-1)_{j+m+n-2}\\
&= (-1)^{m+n}x^{p-1}\sum_{j=0}^{n}(-1)^{j}L(n,j)\mathcal{D}_{j+m+n-2}(1-x)\,,
\end{aligned}$$

which complete the proof. ■

Corollary 11 *For any integer n and any prime number p, there hold*

$$\begin{aligned}
x^n\sum_{k=0}^{p-1}\frac{\mathcal{L}_{n+k}(x)}{k!} &\equiv x^p\sum_{j=0}^{n}(-1)^{n-j}L(n,j)\mathcal{D}_{j+n-1}(1-x)\,,\ n\geq 1,\\
x^n\sum_{k=1}^{p-2}\frac{\mathcal{L}_{n+k}(x)}{(k-1)!} &\equiv x^{p-1}\sum_{j=0}^{n}(-1)^{n-j}L(n,j)\mathcal{D}_{j+n}(1-x)\,,\ n\geq 0.
\end{aligned}$$

Proof. Take $m = 1$ or $m = 2$ in Theorem 10 and use the following congruences: $k!\,(-1)_{p-1-k} \equiv -1$ and $k!\,(-2)_{p-1-k} \equiv k,\ 0 \leq k \leq p-1$. ∎

Remark 1 *Since $\mathcal{D}_1(x) = x$, $\mathcal{D}_2(x) = x^2+1$, then for $n = 0$ or 1 in Corollary 11 we get*

$$x\sum_{k=0}^{p-1}\frac{\mathcal{L}_{k+1}(x)}{k!} \equiv x^p - x^{p+1},\ \sum_{k=1}^{p-1}\frac{\mathcal{L}_k(x)}{(k-1)!} \equiv x^{p-1},\ x\sum_{k=1}^{p-1}\frac{\mathcal{L}_{k+1}(x)}{(k-1)!} \equiv x^{p+2} + x^p.$$

5 Congruences on the $(r_1, \ldots, r_q)$-Bell polynomials

Recall the $\mathbf{r}_q$-Stirling numbers and the $\mathbf{r}_q$-Bell polynomials introduced and studied by Mihoubi et al. [9, 7] can be defined by

$$\begin{aligned}\sum_{j=0}^{n+|\mathbf{r}_{q-1}|}\left\{\begin{matrix} n+|\mathbf{r}_q| \\ j+r_q\end{matrix}\right\}_{\mathbf{r}_q}(x)_j &= (x+r_q)_{r_1}\cdots(x+r_q)_{r_{q-1}}(x+r_q)^n, \\ \mathcal{B}_n(x;\mathbf{r}_q) &= \sum_{j=0}^{n+|\mathbf{r}_{q-1}|}\left\{\begin{matrix} n+|\mathbf{r}_q| \\ j+r_q\end{matrix}\right\}_{\mathbf{r}_q}x^j,\end{aligned}$$

where $\mathbf{r}_q = (r_1, \ldots, r_q)$ and $|\mathbf{r}_q| = r_1 + \cdots + r_q$. Identity (3) gives

$$\mathcal{B}_n(x;\mathbf{r}_q) = (\mathbf{B_x}+r_q)_{r_1}\cdots(\mathbf{B_x}+r_q)_{r_{q-1}}(\mathbf{B_x}+r_q)^n. \tag{8}$$

Let $(u)_{r_1}\cdots(u)_{r_q} = \sum_{k=0}^{|\mathbf{r}_q|}a_k(\mathbf{r}_q)\,u^k$. Then, from (8) and (2) we get

$$x^{r_q}\mathcal{B}_n(x;\mathbf{r}_q) = (\mathbf{B_x})_{r_1}\cdots(\mathbf{B_x})_{r_q}\mathbf{B}_\mathbf{x}^n = \sum_{k=0}^{|\mathbf{r}_q|}a_k(\mathbf{r}_q)\,B_{n+k}(x). \tag{9}$$

Theorem 1.2 of [14] can be generalized as follows

Theorem 12 *Let $n, r_1, \ldots, r_q$ be non-negative integers and p be a prime number such that $r_q \geq \cdots \geq r_1$ and p no divides m. There holds*

$$x^{m+r_q}\sum_{k=1}^{p-1}\frac{B_{n+k}(x;\mathbf{r}_q)}{(-m)^k} \equiv x^p\sum_{j=0}^{n+|\mathbf{r}_{q-1}|}(-1)^{j+r_q+m-1}\left\{\begin{matrix} n+|\mathbf{r}_q| \\ j+r_q\end{matrix}\right\}_{\mathbf{r}_q}\mathcal{D}_{j+r_q+m-1}(1-x).$$

Proof. Use Theorem 1.2 of [14] and Identity (9) to get

$$\begin{aligned} LHS &= x^m\sum_{i=0}^{|\mathbf{r}_q|}a_i(\mathbf{r}_q)\sum_{k=1}^{p-1}\frac{\mathcal{B}_{n+k+i}(x)}{(-m)^k} \\ &\equiv x^p\sum_{i=0}^{|\mathbf{r}_q|}a_i(\mathbf{r}_q)\sum_{j=0}^{n+i}\left\{\begin{matrix} n+i \\ j\end{matrix}\right\}(-1)^{j+m-1}\mathcal{D}_{j+m-1}(1-x)\end{aligned}$$

$$
\begin{aligned}
&= x^p \sum_{i=0}^{|\mathbf{r}_q|} a_i(\mathbf{r}_q) \sum_{j=0}^{n+i} \left\{ {n+i \atop j} \right\} (-1)^{j+m-1} \mathcal{D}_{j+m-1}(1-x) \\
&\equiv x^p \sum_{j=0}^{n+|\mathbf{r}_q|} (-1)^{j+m-1} \left(\sum_{i=0}^{|\mathbf{r}_q|} a_i(\mathbf{r}_q) \left\{ {n+i \atop j} \right\} \right) \mathcal{D}_{j+m-1}(1-x) \\
&\equiv x^p \sum_{j=r_q}^{n+|\mathbf{r}_q|} (-1)^{j+m-1} \left\{ {n+|\mathbf{r}_q| \atop j} \right\}_{\mathbf{r}_q} \mathcal{D}_{j+m-1}(1-x),
\end{aligned}
$$

which is the desired result, and where the last step follows from (9). ■

So, by setting $q=1$, $r_q=r$ in Theorem 12 we may state:

Corollary 13 *For any non-negative integers n, m, r and any prime number $p \nmid m$, we get*

$$
x^{m+r} \sum_{k=1}^{p-1} \frac{\mathcal{B}_{n+k,r}(x)}{(-m)^k} \equiv x^p \sum_{j=0}^{n} \left\{ {n+r \atop j+r} \right\}_r (-1)^{j+m+r-1} \mathcal{D}_{j+m+r-1}(1-x).
$$

In particular, for $x=1$ and $n=0,1$ we get

$$
\begin{aligned}
\sum_{k=1}^{p-1} \frac{\mathcal{B}_{k,r}}{(-m)^k} &\equiv (-1)^{m+r-1} \mathcal{D}_{m+r-1}, \ \ \mathcal{B}_{n,r} = \mathcal{B}_{n,r}(1) \\
\sum_{k=1}^{p-1} \frac{\mathcal{B}_{k+1,r}}{(-m)^k} &\equiv (-1)^{m+r-1} \left(r\mathcal{D}_{m+r-1} - \mathcal{D}_{m+r} \right), \\
\sum_{k=1}^{p-1} \frac{\mathcal{B}_{k+2,r}}{(-m)^k} &\equiv (-1)^{m+r-1} \left(r^2\mathcal{D}_{m+r-1} - (2r+1)\mathcal{D}_{m+r} + \mathcal{D}_{m+r+1} \right).
\end{aligned}
$$

Corollary 14 *Let $n, m, r_q \geq \cdots \geq r_1$ and $s \geq 1$ be non-negative integers, let p be a prime number and let f be a polynomial in $\mathbb{Z}[x]$. Then*

$$
x^{r_q} \mathcal{B}_{n+p^s}(x; \mathbf{r}_q) \equiv x^{r_q} \left(x^p + x^{p^2} + \cdots + x^{p^s} \right) \mathcal{B}_n(x; \mathbf{r}_q) + x^{r_q} \mathcal{B}_{n+1}(x; \mathbf{r}_q).
$$

Proof. By Proposition 3 and (9) we obtain:

$$
\begin{aligned}
x^{r_q} \mathcal{B}_{n+p^s}(x; \mathbf{r}_q) &= x^{r_q} \left(\mathcal{B}_{n+p^s}(x; \mathbf{r}_q) - \mathcal{B}_{n+1}(x; \mathbf{r}_q) \right) + x^{r_q} \mathcal{B}_{n+1}(x; \mathbf{r}_q) \\
&= (\mathbf{B_x})_{r_1} \cdots (\mathbf{B_x})_{r_q} \mathbf{B}_\mathbf{x}^n \left(\mathbf{B}_\mathbf{x}^{p^s} - \mathbf{B_x} \right) + x^{r_q} \mathcal{B}_{n+1}(x; \mathbf{r}_q) \\
&\equiv \left(x^p + x^{p^2} + \cdots + x^{p^s} \right) (\mathbf{B_x})_{r_1} \cdots (\mathbf{B_x})_{r_q} \mathbf{B}_\mathbf{x}^n + x^{r_q} \mathcal{B}_{n+1}(x; \mathbf{r}_q) \\
&= x^{r_q} \left(x^p + x^{p^2} + \cdots + x^{p^s} \right) \mathcal{B}_n(x; \mathbf{r}_q) + x^{r_q} \mathcal{B}_{n+1}(x; \mathbf{r}_q),
\end{aligned}
$$

which give the desired congruence. ■

Other congruences related to the r-Bell polynomials are given as follows:

Theorem 15 *Let $n \geq 0, m \geq 1$ be integers and p be a prime number p. Then*

$$\sum_{r=0}^{p-1} \langle -m \rangle_{p-1-r} x^{m+r} \mathcal{B}_{n,r}(x) \equiv x^{p-1} \sum_{j=0}^{n} (-1)^{j+m} \begin{bmatrix} n \\ j \end{bmatrix} \mathcal{D}_{j+m}(1-x).$$

Proof. By (2) and the congruence $\binom{p-1}{k} \equiv (-1)^k$ we obtain

$$\begin{aligned}
LHS &\equiv x^m \sum_{r=0}^{p-1} \binom{p-1}{r} (m)_{p-1-r} x^r (\mathbf{B_x} + r)^n \\
&= x^m \sum_{r=0}^{p-1} \binom{p-1}{r} (m)_{p-1-r} (\mathbf{B_x})_r \mathbf{B_x}^n \\
&= x^m (\mathbf{B_x} + m)_{p-1} \mathbf{B_x}^n \\
&= (\mathbf{B_x})_{p-1} (\mathbf{B_x})_m (\mathbf{B_x} - m)^n \\
&= x^{p-1} (\mathbf{B_x} + p - 1)_m (\mathbf{B_x} - m + p - 1)^n \\
&\equiv x^{p-1} (\mathbf{B_x} - 1)_m (\mathbf{B_x} - m - 1)^n \\
&= x^{p-1} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (\mathbf{B_x} - 1)_m (\mathbf{B_x} - m - 1)_j \\
&= x^{p-1} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (\mathbf{B_x} - 1)_{j+m} \\
&= x^{p-1} \sum_{j=0}^{n} (-1)^{j+m} \begin{bmatrix} n \\ j \end{bmatrix} \mathcal{D}_{j+m}(1-x).
\end{aligned}$$

So, the desired congruence follows. ■

In particular, for $m = p - 1$ or $p - 2$ in Theorem 15 we get

Corollary 16 *For any integer $n \geq 0$ and any prime p, we obtain*

$$\begin{aligned}
\sum_{r=0}^{p-1} \frac{x^r \mathcal{B}_{n,r}(x)}{r!} &\equiv -\sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} \mathcal{D}_{j+p-1}(1-x), \\
\sum_{r=1}^{p-1} \frac{x^r \mathcal{B}_{n,r}(x)}{(r-1)!} &\equiv -x \sum_{j=0}^{n} (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} \mathcal{D}_{j+p-2}(1-x),
\end{aligned}$$

which yield for $n = 0$: $\sum_{r=0}^{p-1} \frac{x^r}{r!} \equiv -\mathcal{D}_{p-1}(1-x)$, $\sum_{r=1}^{p-1} \frac{x^r}{(r-1)!} \equiv -x\mathcal{D}_{p-2}(1-x)$.

6 Congruences on the Bell numbers without singletons

Now let $\mathbf{B} = \mathbf{B_x} |_{x=1}$ be the Bell umbra and $\mathcal{V}_n$ be the number of ways to partition the set $[n]$ into subsets without singletons. The umbra $\mathbf{B}$ satisfies all the properties of the umbra $\mathbf{B_x}$ with $x = 1$ and $\mathcal{V}_n$ is to be [15]

$$\mathcal{V}_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{B}_k = (\mathbf{B} - 1)^n, \quad \mathcal{B}_n = \mathcal{B}_n(1).$$

Theorem 17 *For any integers $n, m \in \mathbb{N}$ and any prime number p no divides m, we get*

$$\begin{aligned} \sum_{k=0}^{p-1} (-1)^k \mathcal{V}_{n+k} &\equiv \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \mathcal{B}_{j+p-1}, \\ \sum_{k=1}^{p-1} \frac{\mathcal{V}_{n+k}}{(-m)^k} &\equiv (-1)^m \sum_{j=0}^{n} \begin{Bmatrix} n+p-1 \\ j+p-1 \end{Bmatrix}_{p-1} (-1)^j \mathcal{D}_{j+m-2}. \end{aligned}$$

Proof. By $\binom{p-1}{k} \equiv (-1)^k$ and $m^p \equiv m$ we get

$$\begin{aligned} LHS &\equiv \sum_{k=1}^{p-1} \binom{p-1}{k} m^{p-1-k} (\mathbf{B} - 1)^{n+k} \\ &= \left((\mathbf{B} + m - 1)^{p-1} - m^{p-1}\right) (\mathbf{B} - 1)^n \\ &\equiv (\mathbf{B})_{m-1} \left(\mathbf{B}^{p-1} - 1\right) (\mathbf{B} - m)^n. \end{aligned}$$

Then, for $m = 1$ we get

$$\begin{aligned} LHS &\equiv \mathbf{B}^{p-1} (\mathbf{B} - 1)^n - (\mathbf{B} - 1)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \mathbf{B}^{j+p-1} - (\mathbf{B} - 1)^n \\ &\equiv \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \mathcal{B}_{j+p-1} - \mathcal{V}_n. \end{aligned}$$

For $m \geq 2$, by (2), Proposition 3 with $x = 1$ and (3) we get

$$\begin{aligned} LHS &\equiv (\mathbf{B})_{m-1} \left(\mathbf{B}^{p-1} - 1\right) (\mathbf{B} - m)^n \\ &= (\mathbf{B} - 1)_{m-2} (\mathbf{B}^p - \mathbf{B}) (\mathbf{B} - m)^n \\ &\equiv (\mathbf{B} - 1)_{m-2} (\mathbf{B} - m)^n \\ &\equiv (\mathbf{B} - 1)_{m-2} (\mathbf{B} - m + 1 + p - 1)^n \\ &= \sum_{j=0}^{n} \begin{Bmatrix} n+p-1 \\ j+p-1 \end{Bmatrix}_{p-1} (\mathbf{B} - 1)_{m-2} (\mathbf{B} - m + 1)_j \\ &= \sum_{j=0}^{n} \begin{Bmatrix} n+p-1 \\ j+p-1 \end{Bmatrix}_{p-1} (\mathbf{B} - 1)_{j+m-2} \\ &= (-1)^m \sum_{j=0}^{n} \begin{Bmatrix} n+p-1 \\ j+p-1 \end{Bmatrix}_{p-1} (-1)^j \mathcal{D}_{j+m-2}. \end{aligned}$$

So, the desired congruences follows. ■

For $n = 0$ or 1 in Theorem 17 and by Lemma 2.2 of [14] we get

Corollary 18 *For any integers $n, m \in \mathbb{N}$ and any prime number p no divides m, we get*

$$\sum_{k=0}^{p-1} (-1)^k \mathcal{V}_k \equiv \mathcal{B}_{p-1}, \quad \sum_{k=0}^{p-1} (-1)^k \mathcal{V}_{k+1} \equiv \mathcal{B}_p - \mathcal{B}_{p-1} \equiv 2 - \mathcal{B}_{p-1},$$

$$\sum_{k=1}^{p-1} \frac{\mathcal{V}_k}{(-m)^k} \equiv (-1)^m \mathcal{D}_{m-2}, \quad \sum_{k=1}^{p-1} \frac{\mathcal{V}_{k+1}}{(-m)^k} \equiv (-1)^{m+1} (\mathcal{D}_{m-2} + \mathcal{D}_{m-1}), \ m \geq 2.$$

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