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A new class of the r-Stirling numbers and the generalized Bernoulli polynomials

Miloud Mihoubi and Meriem Tiachachat

RECITS laboratory, Faculty of Mathematics,
USTHB, P. O. 32 El Alia 16111 Algiers, Algeria

mmihoubi@usthb.dz, tiachachatmeriem@yahoo.fr

Abstract: The main object of this paper is to express the values at non-negative integers of the generalized Bernoulli polynomials on using a class of the Stirling numbers of the second kind.

Keywords: The quasi-associated r-Stirling numbers; the generalized Bernoulli polynomials.

Résumé : Le but de ce papier est d'exprimer, à l'aide d'une classe de nombres de Stirling de seconde espèce, les valeurs des polynômes de Bernoulli généralisés à arguments entiers non-négatives .

Mots clés : Nombres r-Stirling semi-associés; polynômes de Bernoulli généralisés.

1 Introduction

Recall that, the r-Stirling number of the second kind, $\left\{ {n \atop k} \right\}_r$, counts the number of partitions of the set $[n] := \{1, 2, \ldots, n\}$ into k non-empty subsets such that the elements of the set $[r]$ are in different subsets [3]. These numbers are determined by their generating function

$$\sum_{n\geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{t^n}{n!} = \frac{1}{k!} \left(\exp(t) - 1\right)^k \exp(rt),$$

with $\left\{ {n \atop k} \right\}_1 = \left\{ {n \atop k} \right\}_0 := \left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind.
In [11], the authors expressed $B_n^{(\alpha)}(\pm r)$ in terms of the r-Stirling numbers of both kinds by some formulas, and in [12], they expressed $B_n^{(\alpha)}\left(\pm \frac{r}{m}\right)$ in terms of the r-Witney numbers of both kinds, where, $B_n^{(\alpha)}(x)$ is the n-th high order Bernoulli polynomial (see for example [8, 15]) defined by their exponential generating function to be

$$\sum_{n\geq 0} B_n^{(\alpha)}(x) \frac{t^n}{n!} = \left(\frac{t}{\exp(t) - 1}\right)^\alpha \exp(xt),$$

with $B_n^{(1)}(x) = B_n(x)$ are the classical Bernoulli polynomials.
The generalized Bernoulli polynomials $B_n^{[s-1,\alpha]}(x)$ extend the polynomials introduced by Natalini and Bernardini [13] (see also [6, 2]), and are defined by Kurt [7] (see also [16]) as follows

$$\sum_{n\geq 0} B_n^{[s-1,\alpha]}(x) \frac{t^n}{n!} = \left(\frac{\frac{t^s}{s!}}{\exp(t) - \sum\limits_{j=0}^{s-1} \frac{t^j}{j!}}\right)^\alpha \exp(xt), \quad s \geq 1. \tag{1}$$

In order to give explicit formulas for these polynomials at non-negative integers, we introduce in this paper a class of the r-Stirling numbers of the second kind which can be viewed as a special case of those given in [10].

Definition 1 *For $s \geq 1$, we define the s-quasi-associated r-Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}_r^s$, by the number of partitions of an n-set into k blocks such that the r first elements are in different blocks and each block from the other $k - r$ blocks is of cardinality $\geq s$.*

Below, we show that the numbers $B_n^{[s-1,\alpha]}(r)$ are linked to these numbers (Theorems 6 and 8) by

$$B_n^{[s-1,\alpha]}(r) = \sum_{j=0}^{n} \binom{n+sj}{n, s, \ldots, s}^{-1} \left\{ {n+sj+r \atop j+r} \right\}_r^{s+1} (-\alpha)_j \text{ and}$$

$$B_n^{[s-1,\alpha]}(r) = \alpha \binom{\alpha+n}{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{j!}{\alpha+j} \binom{n+sj}{n, s, \ldots, s}^{-1} \left\{ {n+sj+r \atop j+r} \right\}_r^{s},$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ for $n \geq 1$, $(x)_0 = 1$ and

$$\binom{n+sj}{n, s, \ldots, s} := \frac{(n+sj)!}{n!\,(s!)^j}.$$

Before proving these identities, let us giving some combinatorial properties of the s-quasi-associated r-Stirling numbers of the second kind defined above.

2 Combinatorial properties

From the above definition, we may state that

$$\left\{ n \atop k \right\}_r^s = 0, \quad n < sk \text{ or } k < r,$$

$$\left\{ 0 \atop k \right\}_r^s = \delta_{k,0}, \quad k \geq 0,$$

$$\left\{ n \atop r \right\}_r^s = r^{n-r}, \quad n \geq sr.$$

On using combinatorial arguments, we state these numbers admit an expression given by the following theorem.

Theorem 1 *For $n \geq sk \geq sr \geq 1$, the s-quasi-associated r-Stirling numbers of the second kind can be written as*

$$\left\{ n \atop k \right\}_r^s = \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k = n-r-s(k-r)} \frac{1}{n_1! \cdots n_r!\,(n_{r+1}+s)! \cdots (n_k+s)!}.$$

Proof. To partition a n-set into k blocks $B_1, \ldots, B_k$ such that every block no contains any element of $[r]$ must be of cardinality $\geq s$ and the r first elements are in different blocks (of cardinalities ≥ 1), let the elements of $[r]$ be in different r blocks $B_1, \ldots, B_r$. There is $\frac{1}{(k-r)!}\binom{n-r}{n_1,\ldots,n_k} := \frac{(n-r)!}{(k-r)!}\frac{1}{n_1!\cdots n_k!}$ ways to choose $n_1, \ldots, n_k$ in $[n] \setminus [r]$ such that

- $n_1 \geq 0, \ldots, n_r \geq 0 : n_1, \ldots, n_r$ to be, respectively, in $B_1, \ldots, B_r$,

- $n_{r+1} \geq s, \ldots, n_k \geq s : n_{r+1}, \ldots, n_k$ to be, respectively, in $B_{r+1}, \ldots, B_k$.

Then, the total number of these partitions is

$$\left\{ n \atop k \right\}_r^s = \frac{1}{(k-r)!} \sum_{n_1+\cdots+n_k = n-r,\ n_{r+1} \geq s, \ldots, n_k \geq s} \binom{n-r}{n_1, \ldots, n_k}$$

$$= \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k = n-r-s(k-r)} \frac{1}{n_1! \cdots n_r!\,(n_{r+1}+s)! \cdots (n_k+s)!}.$$

■

By a simple manipulation of Theorem 1, we may state the following:

Corollary 2 *The s-quasi-associated r-Stirling numbers of the second kind have generating function*

$$\sum_{n\geq k}\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r^s \frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{i\geq s}\frac{t^i}{i!}\right)^k \exp\left(rt\right).$$

Upon using combinatorial arguments, we give below three recurrence relations.

Proposition 3 *For $n > sr \geq 1$ we have*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r^s = k\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_r^s + \binom{n-r-1}{s-1}\left\{ \begin{matrix} n-s \\ k-1 \end{matrix} \right\}_r^s.$$

Proof. To partition the set $[n]$ into k blocks such that every block no intersect $[r]$ must be of cardinality $\geq s$ and the elements of $[r]$ are in different blocks, we separate the element n and proceed as follows:
(a) If n is in a block intersecting $[r]$, there is $\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r^s$ ways to partition the set $[n-1]$ into k blocks with the same conditions, the element n (not really used) can be inserted in the r blocks which have intersection with $[r]$, so we count $r\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r^s$ ways.
(b) If n is in a block of cardinality s and no intersect $[r]$, there is $\binom{n-r-1}{s-1}$ ways to choose $s-1$ elements to be with this element in the same block. The remaining $n-s$ elements can be partitioned into $k-1$ blocks in $\left\{ \begin{smallmatrix} n-s \\ k-1 \end{smallmatrix} \right\}_r^s$ ways. So, the number of ways in this case must be $\binom{n-r-1}{s-1}\left\{ \begin{smallmatrix} n-s \\ k-1 \end{smallmatrix} \right\}_r^s$.
(c) If n is in a block of cardinality $\geq s+1$ and no intersect $[r]$, there is $(k-r)\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r^s$ ways.
Then, the number of all partitions is given by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r^s = r\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r^s + \binom{n-r-1}{s-1}\left\{ \begin{smallmatrix} n-s \\ k-1 \end{smallmatrix} \right\}_r^s + (k-r)\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_r^s$.
■

Proposition 4 *For $r \geq 2$ we have*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r^s = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{r-1}^s - (r-1)\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}_{r-1}^s.$$

Proof. The number of partitions of the set $[n]$ into k blocks such that every block no intersect $[r]$ must be of cardinality $\geq s$ and the elements of $[r]$ are in different blocks but not the element r, there is $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r-1}^s - \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_r^s$ ways. This number can obtained by considering the element r to be in one of the $r-1$ blocks intersecting $[r-1]$, which gives $(r-1)\left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}_{r-1}^s$ ways. ■

Proposition 5 *For $r \geq 1$ we have*

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r^s = \sum_{j\geq 0}\binom{n-r}{j}\left\{ \begin{matrix} n-1-j \\ k-1 \end{matrix} \right\}_{r-1}^s.$$

Proof. To partition the set $[n]$ into k blocks such that every block no intersect $[r]$ must be of cardinality $\geq s$ and the elements of $[r]$ are in different blocks, the element r can be in a block of cardinality $j+1$ in $\binom{n-r}{j}\left\{ {n-1-j \atop k-1} \right\}_{r-1}^{s}$ ways:
(a) $\binom{n-r}{j}$ is the number of ways for choosing the j elements between $(n-1)-(r-1)$ elements of $[n]\setminus[r]$ to be in the same block with the element r and
(b) $\left\{ {n-1-j \atop k-1} \right\}_{r-1}^{s}$ is the number of ways to partition the remained $n-(j+1)=n-1-j$ elements into $k-1$ blocks such that every block no intersect $[r]$ must be of cardinality $\geq s$ and the elements of $[r-1]$ are in different blocks. ■

3 Application to the generalized Bernoulli polynomials

We give in this section two expressions in terms of the s-quasi-associated r-Stirling numbers of the second kind for $B_n^{[s-1,\alpha]}(r)$. The following theorem gives a simplified expression for $B_n^{[s-1,\alpha]}(r)$ for all non-negative integers r.

Theorem 6 *We have*

$$B_n^{[s-1,\alpha]}(r)=\sum_{j=0}^{n}\binom{n+sj}{n,s,\ldots,s}^{-1}\left\{ {n+sj+r \atop j+r} \right\}_{r}^{s+1}(-\alpha)_j$$

and for $\alpha=-k$ be a non-positive integer, we have

$$B_n^{[s-1,-k]}(r)=k!\binom{n+sk}{n,s,\ldots,s}^{-1}\left\{ {n+sk+r \atop k+r} \right\}_{r}^{s},$$

Proof. From the definition of $B_n^{[s-1,\alpha]}(x)$ we get

$$\sum_{n\geq 0}B_n^{[s-1,\alpha]}(r)\frac{t^n}{n!}=\left(\sum_{j=0}^{\infty}\binom{j+s}{s}^{-1}\frac{t^j}{j!}\right)^{-\alpha}\exp(rt)=\exp(rt)\sum_{n\geq 0}f_n(-\alpha)\frac{t^n}{n!},$$

which gives $B_n^{[s-1,\alpha]}(r)=\sum\limits_{k=0}^{n}\binom{n}{k}r^{n-k}f_k(-\alpha)$, where $(f_n(x))$ is a sequence of binomial type with $f_n(1)=\binom{n+s}{s}^{-1}$. Use the known relation (see [14])

$$f_n(-\alpha)=\sum_{j=0}^{n}B_{n,j}\left(\binom{i+s}{s}^{-1}\right)(-\alpha)_j$$

to obtain

$$B_n^{[s-1,\alpha]}(r)=\sum_{j=0}^{n}(-\alpha)_j\sum_{k=j}^{n}\binom{n}{k}r^{n-k}B_{k,j}\left(\binom{i+s}{s}^{-1}\right),$$

where $B_{n,k}(x_i) := B_{n,k}(x_1, x_2, ...)$ is the partial Bell polynomial, see [1, 4, 9]. Now, the exponential generating function of

$$A(n,j) := \sum_{k=j}^{n} \binom{n}{k} r^{n-k} B_{k,j}\left(\binom{i+s}{s}^{-1}\right)$$

must be

$$\begin{aligned}
\sum_{n\geq j} A(n,j)\frac{t^n}{n!} &= \exp(rt)\sum_{k\geq j} B_{k,j}\left(\binom{i+s}{s}^{-1}\right)\frac{t^k}{k!} \\
&= \frac{1}{j!}\left(\sum_{i\geq 1}\frac{s!t^i}{(i+s)!}\right)^j \exp(rt) \\
&= \frac{t^{-sj}}{j!}\left(s!\sum_{i\geq s+1}\frac{t^i}{i!}\right)^j \exp(rt) \\
&= \sum_{n\geq j}\frac{(s!)^j n!}{(n+sj)!}\left\{\begin{matrix} n+sj+r \\ j+r \end{matrix}\right\}_r^{s+1}\frac{t^n}{n!}.
\end{aligned}$$

So, we obtain

$$A(n,j) = \frac{n!(s!)^j}{(n+sj)!}\left\{\begin{matrix} n+sj+r \\ j+r \end{matrix}\right\}_r^{s+1} \quad \text{and} \quad B_n^{[s-1,\alpha]}(r) = \sum_{j=0}^{n} A(n,j)(-\alpha)_j.$$

The second part of the theorem follows from the expansion

$$\sum_{n\geq 0} B_n^{[s-1,-k]}(r)\frac{t^n}{n!} = t^{-sk}\left(\sum_{i\geq s}\frac{t^i}{i!}\right)^k \exp(rt) = k!t^{-sk}\sum_{n\geq sk}\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r^s \frac{t^n}{n!}.$$

■

In particular, for $s = 1$, we get

Corollary 7 *We have*

$$B_n^{(\alpha)}(r) := B_n^{[0,\alpha]}(r) = \sum_{j=0}^{n}\frac{n!}{(n+j)!}\left\{\begin{matrix} n+j+r \\ j+r \end{matrix}\right\}_r^2 (-\alpha)_j.$$

So, the values of the Bernoulli polynomials at non-negative integers is given by

$$B_n(r) := B_n^{(1)}(r) = \sum_{j=0}^{n}(-1)^j\binom{n+j}{j}^{-1}\left\{\begin{matrix} n+j+r \\ j+r \end{matrix}\right\}_r^2,$$

and from the known identity $B_n^{(n+1)}(x) = (x-1)^{\underline{n}}$, we may state that for $\alpha = n+1$ we have

$$\sum_{j=0}^{n} (-1)^j \begin{Bmatrix} n+j+r \\ j+r \end{Bmatrix}_r^2 = (r-1)^{\underline{n}}.$$

Other expressions in terms of the s-quasi-associated r-Stirling numbers of the second kind for $B_n^{[s-1,\alpha]}(r)$ are given as follows.

Theorem 8 *Let p, r, n, s be non-negative integers such that $s \geq 1$, $p \geq n$. Then, we have*

$$B_n^{[s-1,\alpha]}(r) = \alpha \binom{\alpha+p}{p} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \frac{j!}{\alpha+j} \binom{n+sj}{n,s,\ldots,s}^{-1} \begin{Bmatrix} n+sj+r \\ j+r \end{Bmatrix}_r^s,$$

where

$$\binom{x}{k} := \frac{(x)_k}{k!}.$$

Proof. For any polynomial f of degree $n \leq p$, the Melzak's formula [5] gives

$$f(x+\alpha) = \alpha \binom{\alpha+p}{p} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \frac{f(x-j)}{\alpha+j},$$

where $\binom{x}{n} = \frac{x^{\underline{n}}}{n!}$. By Theorem 6, we deduce that $B_n^{[s-1,\alpha]}(x)$ is a polynomial in α of degree $\leq n$, then by setting $f(x) = B_n^{[s-1,x]}(r)$ in the Melzak's Formula we obtain

$$B_n^{[s-1,\alpha]}(r) = \alpha \binom{\alpha+p}{p} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \frac{B_n^{[s-1,-j]}(r)}{\alpha+j}.$$

The desired identity follows by using the second identity of Theorem 6. ∎

For the choice $p = n$ in Theorem 8, we get

Corollary 9 *We have*

$$B_n^{[s-1,\alpha]}(r) = \alpha \binom{\alpha+n}{n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{j!}{\alpha+j} \binom{n+sj}{n,s,\ldots,s}^{-1} \begin{Bmatrix} n+sj+r \\ j+r \end{Bmatrix}_r^s.$$

This gives the values of the high order Bernoulli polynomials at non-negative integers to be

$$B_n^{(\alpha)}(r) := B_n^{[0,\alpha]}(r) = \alpha \binom{\alpha+n}{n} \sum_{j=0}^{n} \frac{(-1)^j}{\alpha+j} \frac{\binom{n}{j}}{\binom{n+j}{j}} \begin{Bmatrix} n+j+r \\ j+r \end{Bmatrix}_r.$$

So, the values of the Bernoulli polynomials at non-negative integers are given by

$$B_n(r) = \sum_{j=0}^{n} (-1)^j \frac{\binom{n+1}{j+1}}{\binom{n+j}{j}} \begin{Bmatrix} n+j+r \\ j+r \end{Bmatrix}_r.$$

References

[1] E. T. Bell, Exponential polynomials, *Ann. Math.,* 35 (1934) 258-277.

[2] G. Bretti, P. Natalini, P. E. Ricci, Generalizations of the Bernoulli and Appell polynomials, *Abst. Appl. Anal.,* 7 (2004) 613–623.

[3] A. Z. Broder, The r-Stirling numbers, *Discrete Math.*, 49 (1984) 241–259.

[4] L. Comtet, *Advanced Combinatorics,* D. Reidel Publishing Company, Dordrecht-Holland / Boston-U.S.A, (1974).

[5] H. W. Gould, Higher order extensions of Melzak's formula, *Util. Math.,* 72 (2007) 23–32.

[6] A. Hassen and H. D. Nguyen, Hypergeometric Bernoulli polynomials and Appell sequences, *Int. J. Number Theory*, 4 (5) (2008) 767–774.

[7] B. Kurt, A further generalization of the Bernoulli polynomials and on the $2D$-Bernoulli polynomials $B_n^2(x, y)$, *Appl. Math. Sci.,* 4 (2010) 2315–2322.

[8] Y. L. Luke, The special functions and their approximations, vol. I, Academic Press, New York, London, 1969.

[9] M. Mihoubi, Bell polynomials and binomial type sequences, *Discrete Math.*, 308 (2008) 2450–2459.

[10] M. Mihoubi and M. Rahmani, The partial r-Bell polynomials, *Afr. Mat.*, 28 (Issue 7-8) (2017) 1167–1183.

[11] M. Mihoubi and M. Tiachachat, The values of the high order Bernoulli polynomials at integers and the r-Stirling numbers, available at http://arxiv.org/abs/1401.5958.

[12] M. Mihoubi and M. Tiachachat, Some applications of the r-Whitney numbers, *C. R. Acad. Sci. Paris, Ser.I*, 352 (2014) 965–969.

[13] P. Natalini, A. Bernardini, A generalization of the Bernoulli polynomials, *J. Appl. Math.,* 3 (2003) 155–163.

[14] S. Roman, The umbral calculus, Academic Press, INC, 1984.

[15] H. M. Srivastava, An explicit formula for the generalized Bernoulli polynomials, *J. Math. Anal. Appl.,* 130 (1988) 509–513.

[16] R. Tremblay, S. Gaboury, B. J. Fugère, A new class of generalized Apostol–Bernoulli polynomials and some analogues of the Srivastava–Pintér addition theorem, *Appl. Math. Lett.,* 24 (2011) 1888–1893.