Les Annales RECITS (http://www.lrecits.usthb.dz)
Vol. 04, 2017, pages 13-19

Multipartitional polynomials: Combinatorial and probabilistic interpretations

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Abstract: In this paper, we establish a combinatorial interpretation and a probabilistic interpretation for multipartitional polynomials.

Keywords: Multipartitional polynomials; combinatorial interpretation; probabilistic interpretation.

Résumé : Dans ce papier, on établit une interprétation combinatoire et une interprétation probabiliste pour les polynômes multipartitionnels.

Mots clés : Polynômes multipartitionnels; interprétation combinatoire; interprétation probabiliste.

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1 Introduction and background material

We can find an introduction and some developments of the partial and complete Bell polynomials in [1, 4, 6, 7, 8, 13, 15, 16]. An extension to the partial and complete bipartitional polynomials are developed in [3, 5, 10]. For the generalized case, the partial and complete multipartitional polynomials are considered by the authors in [2, 9, 11], see also [14]. For example, ones of their applications appear in Tutte polynomials and the inverse relations, see [9, 14]. The large application of the multipartitional polynomials gives a motivation to develop this mathematical tool. In precedent works [2, 11], the authors established identities on multipartitional polynomials. They gave some results concerning the preservation of sequences of multinomial type and some other results. In this paper, we give combinatorial and probabilistic interpretations for these polynomials. Let us introduce, as in [2, 11], some notation, definitions and the background material. For $m, m_1, \ldots, m_r$ integers such that $\sum_{i=1}^{r} m_i = m$, we define

$$\binom{m}{m_1, \ldots, m_r} = \begin{cases} \frac{m!}{m_1! \cdots m_r!} & \text{if } m_i \geq 0, \ i = 1, \ldots, r \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{R}$ be the set of real numbers.
Below, we use the following notation:

- for $\underline{\mathbf{a}} = (a_1, \ldots, a_r) \in \mathbb{R}^r$, $\underline{\mathbf{b}} = (b_1, \ldots, b_r) \in \mathbb{R}^r$ we set

  $\underline{\mathbf{a}}.\underline{\mathbf{b}} = a_1 b_1 + \cdots + a_r b_r$, $\underline{\mathbf{a}} + \underline{\mathbf{b}} = (a_1 + b_1, \ldots, a_r + b_r)$, $\lambda \underline{\mathbf{a}} = (\lambda a_1, \ldots, \lambda a_r)$,

  $(\underline{\mathbf{a}} \geq \underline{\mathbf{b}}) \Leftrightarrow (a_1 \geq b_1, \ldots, a_r \geq b_r)$, $(\underline{\mathbf{a}} > \underline{\mathbf{b}}) \Leftrightarrow (a_1 > b_1, \ldots, a_r > b_r)$,

  $\varphi(\underline{\mathbf{a}}, \underline{\mathbf{b}}) = 1$ if $\underline{\mathbf{a}} \geq \underline{\mathbf{b}}$ and 0 otherwise,

- for $\underline{\mathbf{n}} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we set

  $\underline{\mathbf{t}}^{\underline{\mathbf{n}}} = t_1^{n_1} \cdots t_r^{n_r}$, $\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!} = \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}$, $\underline{\mathbf{n}}! = n_1! \cdots n_r!$, $|\underline{\mathbf{n}}| = n_1 + \cdots + n_r$, $\underline{\mathbf{1}} = (1, \ldots, 1)$,

  $\prod_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}} (k_{\underline{\mathbf{i}}}!) = \prod_{i_r=0}^{n_r} \cdots \prod_{i_1=0}^{n_1} (k_{i_1, \cdots, i_r}!)$, $k_{\underline{\mathbf{i}}} = k_{i_1, \cdots, i_r}$ with the convention $k_{\underline{\mathbf{0}}} = 0$,

  $\binom{\underline{\mathbf{n}}}{\underline{\mathbf{j}}} = \frac{\underline{\mathbf{n}}!}{\underline{\mathbf{j}}!(\underline{\mathbf{n}} - \underline{\mathbf{j}})!}$, $\underline{\mathbf{n}}, \underline{\mathbf{j}} \in \mathbb{N}^r$, $\binom{\underline{\mathbf{n}}}{\underline{\mathbf{j}}_1, \cdots, \underline{\mathbf{j}}_k} = \frac{\underline{\mathbf{n}}!}{\underline{\mathbf{j}}_1! \cdots \underline{\mathbf{j}}_k!}$, $\underline{\mathbf{n}}, \underline{\mathbf{j}}_1, \cdots, \underline{\mathbf{j}}_k \in \mathbb{N}^r$ with $\underline{\mathbf{j}}_1 + \cdots + \underline{\mathbf{j}}_k = \underline{\mathbf{n}}$,

- for $(m, n) \in \mathbb{N}^2$ we set

  $B_{m,n;k}(x_{i,j}) = B_{m,n;k}(x_{0,1}, x_{1,0}, \cdots, x_{0,i+j}, x_{1,i+j-1}, \ldots, x_{i+j-1,1}, x_{i+j,0}, \cdots)$,

  $A_{m,n}(x_{i,j}) = A_{m,n}(x_{0,1}, x_{1,0}, \cdots, x_{0,i+j}, x_{1,i+j-1}, \ldots, x_{i+j-1,1}, x_{i+j,0}, \cdots)$,

  and for $\underline{\mathbf{n}} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we set

  $B_{\underline{\mathbf{n}},k}(x_{\underline{\mathbf{i}}}) = B_{n_1, \ldots, n_r, k}(x_{0,\ldots,0,1}, \ldots, x_{1,0,\ldots,0}, x_{0,\ldots,0,2}, x_{0,\ldots,0,1,1}, \ldots)$, $\underline{\mathbf{n}} \geq \underline{\mathbf{0}}$, $k \geq 0$ and $= 0$ otherwise,

  $A_{\underline{\mathbf{n}}}(x_{\underline{\mathbf{i}}}) = A_{n_1, \ldots, n_r}(x_{0,\ldots,0,1}, \ldots, x_{1,0,\ldots,0}, x_{0,\ldots,0,2}, x_{0,\ldots,0,1,1}, \ldots)$, $\underline{\mathbf{n}} \geq \underline{\mathbf{0}}$ and $= 0$ otherwise,

  in the $B_{n_1, \ldots, n_r, k}$ and $A_{n_1, \ldots, n_r}$, the variable $x_{n_1, \ldots, n_r}$ is before $x_{m_1, \ldots, m_r}$ if $|\underline{\mathbf{n}}| < |\underline{\mathbf{m}}|$ and when $|\underline{\mathbf{n}}| = |\underline{\mathbf{m}}|$ we use the lexicographical order,

- for $\underline{\mathbf{n}} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we set $f_{\underline{\mathbf{n}}}(x) = f_{n_1,\ldots,n_r}(x)$ if $\underline{\mathbf{n}} \geq \underline{\mathbf{0}}$, $f_{\underline{\mathbf{0}}}(x) = 1$ and $= 0$ otherwise,

The complete (exponential) multipartitional polynomials $A_{\underline{\mathbf{n}}}$ in the variables $(x_{\underline{\mathbf{i}}},\ \underline{\mathbf{i}} \neq \underline{\mathbf{0}})$, are defined by the sum

$$A_{\underline{\mathbf{n}}}(x_{\underline{\mathbf{i}}}) := \sum \frac{\underline{\mathbf{n}}!}{\prod_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}}(k_{\underline{\mathbf{i}}}!)} \prod_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}} \left(\frac{x_{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\right)^{k_{\underline{\mathbf{i}}}}, \quad k_{\underline{\mathbf{0}}} = 0, \tag{1}$$

where the summation is extended over all partitions of the multipartite number $\underline{\mathbf{n}} = (n_1, \ldots, n_r)$, that is, over all nonnegative integers $(k_{\underline{\mathbf{i}}},\ \underline{\mathbf{0}} \leq \underline{\mathbf{i}} \leq \underline{\mathbf{n}})$ solution of the equations

$$\sum_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}} i_j k_{\underline{\mathbf{i}}} = n_j \quad (j = 1, \ldots, r). \tag{2}$$

The partial (exponential) multipartitional polynomials $B_{\underline{\mathbf{n}},k}$ with the variables $(x_{\underline{\mathbf{i}}},\ \underline{\mathbf{i}} \neq \underline{\mathbf{0}})$, of degree k, are defined by the sum

$$B_{\underline{\mathbf{n}};k}(x_{\underline{\mathbf{i}}}) := \sum \frac{\underline{\mathbf{n}}!}{\prod_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}}(k_{\underline{\mathbf{i}}}!)} \prod_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}} \left(\frac{x_{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\right)^{k_{\underline{\mathbf{i}}}}, \quad k_{\underline{\mathbf{0}}} = 0, \tag{3}$$

where the summation is extended over all partitions of the multipartite number $\underline{\mathbf{n}} = (n_1, \ldots, n_r)$ into k parts, that is, over all nonnegative integers $(k_{\underline{\mathbf{i}}},\ \underline{\mathbf{0}} \leq \underline{\mathbf{i}} \leq \underline{\mathbf{n}})$ solution of the equations

$$\sum_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}} k_{\underline{\mathbf{i}}} = k, \quad \sum_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}} i_j k_{\underline{\mathbf{i}}} = n_j \quad (j = 1, \ldots, r), \quad k_{\underline{\mathbf{0}}} = 0. \tag{4}$$

From the above definitions, one can verify that the exponential generating function of $A_{n_1,\ldots,n_r}$ is given by

$$\sum_{\underline{\mathbf{n}} \geq \underline{\mathbf{0}}} A_{\underline{\mathbf{n}}}\left(x_{\underline{\mathbf{j}}}\right) \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!} = \exp\left(\sum_{|\underline{\mathbf{i}}| \geq 1} x_{\underline{\mathbf{i}}} \frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\right), \tag{5}$$

and the exponential generating function for $B_{\underline{\mathbf{n}};k}$ is given by

$$\sum_{|\underline{\mathbf{n}}| \geq k} B_{\underline{\mathbf{n}};k}\left(x_{\underline{\mathbf{j}}}\right) \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!} = \frac{1}{k!}\left(\sum_{|\underline{\mathbf{i}}| \geq 1} x_{\underline{\mathbf{i}}} \frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\right)^k. \tag{6}$$

2 Combinatorial and probabilistic interpretations

Let $x_1, x_2, \ldots$ be nonnegative integers. It is known that the partial Bell polynomial $B_{n,k}(x_1, x_2, ...)$ enumerate the number of (colored) partitions of a n-set into k parts such that any block of length i can be colored with x_i colors, see [6, 12].

Generally, in the multivariate case we suggest the following combinatorial interpretation.

Theorem 1 *Let $(x_{\underline{\mathbf{m}}};\underline{\mathbf{m}}\in\mathbb{N}^r)$ be a sequence with r index of nonnegative integers, with $x_{\underline{\mathbf{0}}}=0$, and $\underline{\mathbf{n}}=(n_1,\ldots,n_r)\in\mathbb{N}^r$. Then the partial multipartitional polynomial $B_{\underline{\mathbf{n}};k}(x_{\underline{\mathbf{i}}})$ enumerate the number of partitions of a set $T=T_1\cup\cdots\cup T_r$ with $|T_1|=n_1,\ldots,|T_r|=n_r$ into k parts such that any part of length $|\underline{\mathbf{i}}|=i_1+\cdots+i_r$ formed by i_1 elements of $T_1,\ldots,$ i_r elements of T_r, can be colored with $x_{\underline{\mathbf{i}}}=x_{i_1,\ldots,i_r}$ colors.*

Proof. Given a block $j\in\{1,\ldots,k\}$ of $\left|\underline{\mathbf{m}}_j\right|=m_{1j}+\cdots+m_{rj}\,(\geq 1)$ elements of T which whose elements are formed by m_{1j} elements of $T_1,\ldots,$ m_{rj} elements of T_r, and can be colored with $x_{\underline{\mathbf{m}}_j}=x_{m_{1j},\ldots,m_{rj}}$colors. To form a partition, we may choose k blocks, that is in $\frac{1}{k!}\binom{n_1}{m_{11},\ldots,m_{1k}}\cdots\binom{n_r}{m_{r1},\ldots,m_{rk}}$ ways, which can be colored by $x_{m_{11},\ldots,m_{r1}}\cdots x_{m_{1k},\ldots,m_{rk}}$ colors.

Then, the total number of colored partitions is

$$N_{\underline{\mathbf{n}};k}=\frac{1}{k!}\sum\binom{n_1}{m_{11},\ldots,m_{1k}}\cdots\binom{n_r}{m_{r1},\ldots,m_{rk}}x_{m_{11},\ldots,m_{r1}}\cdots x_{m_{1k},\ldots,m_{rk}},$$

where the summation is under of all nonnegative integers m_{ij} satisfaying $\sum_{j=1}^{k}m_{ij}=n_j,\ \ i=1,\ldots,r$. Hence

$$\begin{aligned}\sum_{\underline{\mathbf{n}}\geq\underline{\mathbf{0}}}N_{\underline{\mathbf{n}};k}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}&=\frac{1}{k!}\sum_{n_1,\cdots,n_r\geq 0}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\sum\binom{n_1}{m_{11},\ldots,m_{1k}}\cdots\binom{n_r}{m_{r1},\ldots,m_{rk}}\\&\quad\times x_{m_{11},\ldots,m_{r1}}\cdots x_{m_{1k},\ldots,m_{rk}}\\&=\frac{1}{k!}\sum_{m_{ij}\geq 0}\frac{t_1^{\sum_{j=1}^{k}m_{1j}}\cdots t_r^{\sum_{j=1}^{k}m_{rj}}}{\prod_{i,j}(m_{ij}!)}x_{m_{11},\ldots,m_{r1}}\cdots a_{m_{1k},\ldots,m_{rk}}\end{aligned}$$

and this can written as

$$\sum_{\underline{\mathbf{n}}\geq\underline{\mathbf{0}}}N_{\underline{\mathbf{n}};k}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}=\frac{1}{k!}\prod_{j=1}^{k}\left(\sum_{m_{1j},\ldots,m_{rj}\geq 0}x_{m_{1j},\ldots,m_{rj}}\frac{t_1^{m_{1j}}\cdots t_r^{m_{rj}}}{m_{1j}!\ldots m_{rj}!}\right)=\frac{1}{k!}\left(\sum_{i_1,\ldots,i_r\geq 0}x_{i_1,\ldots,i_r}\frac{t_1^{i_1}\cdots t_r^{i_r}}{i_1!\ldots i_r!}\right)^k,$$

which implies via (6) that $N_{\underline{\mathbf{n}};k}=B_{\underline{\mathbf{n}};k}(x_{\underline{\mathbf{i}}})$. ■

Link between the partial Bell polynomials and discrete probability is given as follows: Let $\{X_n\}$ be a sequence of independent discrete random variables with the same law of probability $p_j:=P(X_1=j)$, $\ j\geq 0$ and let $S_k:=X_1+\cdots+X_k$. Then we have

$$P(S_k=n)=\frac{k!}{(n+k)!}B_{n+k;k}(1!p_0,\ldots,m!p_{m-1},\ldots).$$

Link between the partial Bell polynomials and the moments of a random variable is given as follows:
Let $\{X_n\}$ be a sequence of independent random variables with all its moments exist and are the same, $\mu_n=\mathrm{E}(X^n)$ and let $S_k:=X_1+\cdots+X_k$. Then we have

$$\mathrm{E}(S_k^n)=\binom{n+k}{k}^{-1}B_{n+k;k}(1,2\mu_1,\ldots,m\mu_{m-1},\ldots).$$

For the above probalistic interpretations, one can see [12].
The above results become in the multivariate case as follows:

Theorem 2 *Let* $\mathbf{X}^{(1)} = \left(X_1^{(1)}, \ldots, X_r^{(1)}\right), \ldots, \mathbf{X}^{(k)} = \left(X_1^{(k)}, \ldots, X_r^{(k)}\right)$ *be* k *independent random vectors with the same discrete law of probability*

$$P\left(\mathbf{X}^{(j)} = \underline{\mathbf{i}}\right) = \frac{p_{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}, \quad \underline{\mathbf{i}} = (i_1, \ldots, i_r) \in \mathbb{N}^r, \ j = 1, \ldots, k$$

and set

$$U_i^{(k)} = X_i^{(1)} + \cdots + X_i^{(k)}, \quad i = 1, \ldots, r, . \ k = 1, 2, \ldots,$$
$$\mathbf{U}^{(k)} = \left(U_1^{(k)}, \ldots, U_r^{(k)}\right).$$

Then, we have

$$P\left(\mathbf{U}^{(k)} = \underline{\mathbf{n}}\right) = \frac{k!}{\left(\underline{\mathbf{n}} + k\underline{\mathbf{e}}_f\right)!} B_{\underline{\mathbf{n}}+k\underline{\mathbf{e}}_f;k}\left(\left(\underline{\mathbf{e}}_f.\underline{\mathbf{i}}\right) p_{\underline{\mathbf{i}}-\underline{\mathbf{e}}_f}\right), \quad f = 1, 2, \ldots, r,$$

where $\underline{\mathbf{e}}_f = \left(0, \ldots, 0, \overset{\textit{position } f}{1}, 0, \ldots, 0\right)$.

Proof. We have

$$\Omega_{\mathbf{X}^{(j)}}\left(\underline{\mathbf{t}}\right) = \mathrm{E}\left(t_1^{X_1^{(j)}} \cdots t_r^{X_r^{(j)}}\right) = \sum_{\underline{\mathbf{n}} \geq \underline{\mathbf{0}}} p_{\underline{\mathbf{n}}} \underline{\mathbf{t}}^{\underline{\mathbf{n}}}, \quad \underline{\mathbf{t}} = (t_1, \ldots, t_r),$$

$$\Omega_{\mathbf{U}^{(k)}}\left(\underline{\mathbf{t}}\right) = \mathrm{E}\left(t_1^{U_1^{(k)}} \cdots t_r^{U_r^{(k)}}\right) = \prod_{j=1}^{k} \mathrm{E}\left(t_1^{X_1^{(j)}} \cdots t_r^{X_r^{(j)}}\right) = \left(\Pi_{\mathbf{X}^{(1)}}\left(\mathbf{t}\right)\right)^k = \left(\sum_{\underline{\mathbf{n}} \geq \underline{\mathbf{0}}} p_{\underline{\mathbf{n}}} \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}\right)^k.$$

Let $n_f = \underline{\mathbf{e}}_f.\underline{\mathbf{n}}$ be the f-th conponent of a vector $\underline{\mathbf{n}}$. Then

$$t_f^k \Omega_{\mathbf{U}^{(k)}}\left(\underline{\mathbf{t}}\right) = \left(\sum_{\underline{\mathbf{n}} \geq \underline{\mathbf{e}}_f} n_f p_{\underline{\mathbf{n}}-\underline{\mathbf{e}}_f} \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}\right)^k = k! \sum_{\underline{\mathbf{n}} \geq k\underline{\mathbf{e}}_f} B_{\underline{\mathbf{n}};k}\left(i_f p_{\underline{\mathbf{i}}-\underline{\mathbf{e}}_f}\right) \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!},$$
$$t_f^k \Omega_{\mathbf{U}^{(k)}}\left(\underline{\mathbf{t}}\right) = \sum_{\underline{\mathbf{n}} \geq k\underline{\mathbf{e}}_f} P\left(\mathbf{U}^{(k)} = \underline{\mathbf{n}} - k\underline{\mathbf{e}}_f\right) \underline{\mathbf{t}}^{\underline{\mathbf{n}}},$$

which imply the desired identity. ■

Theorem 3 *Let* $\mathbf{X}^{(1)} = \left(X_1^{(1)}, \ldots, X_r^{(1)}\right), \ldots, \mathbf{X}^{(k)} = \left(X_1^{(k)}, \ldots, X_r^{(k)}\right)$ *be* k *independant random vectors with the same probability distribution and*

$$\mu_{\underline{\mathbf{n}}} = \mathrm{E}\left(X_1^{n_1} \ldots X_r^{n_r}\right), \qquad \underline{\mathbf{n}} = (n_1, \cdots, n_r) \in \mathbb{N}^r$$

and set

$$U_j^{(k)} := X_j^{(1)} + \cdots + X_j^{(k)}, \quad j = 1, \ldots, r, . \ k = 1, 2, \ldots,$$
$$\mathbf{U}^{(k)} = \left(U_1^{(k)}, \ldots, U_r^{(k)}\right).$$

Then, we have

$$\mathrm{E}\left(\left(U_1^{(k)}\right)^{n_1}\cdots\left(U_r^{(k)}\right)^{n_r}\right)=\binom{(\underline{\mathbf{e}}_f\cdot\underline{\mathbf{n}})+k}{k}^{-1}B_{\underline{\mathbf{n}}+k\underline{\mathbf{e}}_f;k}\left((\underline{\mathbf{e}}_f\cdot\underline{\mathbf{i}})\,\mu_{\underline{\mathbf{i}}-\underline{\mathbf{e}}_f}\right).$$

Proof. We have

$$\begin{aligned}\Phi_{\mathbf{X}^{(j)}}(\underline{\mathbf{t}})&=\mathrm{E}\left(\exp\left(t_1X_1^{(j)}+\cdots+t_rX_r^{(j)}\right)\right)\\&=\sum_{\underline{\mathbf{n}}\geq\underline{\mathbf{0}}}\mu_{\underline{\mathbf{n}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!},\\\Phi_{\mathbf{U}^{(k)}}(\underline{\mathbf{t}})&=\mathrm{E}\left(\exp\left(t_1U_1^{(k)}+\cdots+t_rU_r^{(k)}\right)\right)\\&=\prod_{j=1}^{k}\mathrm{E}\left(\exp\left(t_1X_1^{(j)}+\cdots+t_rX_r^{(j)}\right)\right)\\&=\left(\Phi_{\mathbf{X}^{(1)}}(\underline{\mathbf{t}})\right)^k.\end{aligned}$$

These imply that

$$\begin{aligned}t_f^k\Phi_{\mathbf{U}^{(k)}}(\underline{\mathbf{t}})&=t_f^k\left(\sum_{\underline{\mathbf{n}}\geq\underline{\mathbf{0}}}\mu_{\underline{\mathbf{n}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}\right)^k=\left(\sum_{\underline{\mathbf{n}}\geq\underline{\mathbf{0}}}n_f\mu_{\underline{\mathbf{n}}-\underline{\mathbf{e}}_f}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}\right)^k=k!\sum_{\underline{\mathbf{n}}\geq\underline{\mathbf{0}}}B_{\underline{\mathbf{n}};j}\left(i_f\mu_{\underline{\mathbf{i}}-\underline{\mathbf{e}}_f}\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!},\\t_f^k\Phi_{\mathbf{U}^{(k)}}(\underline{\mathbf{t}})&=\sum_{\underline{\mathbf{n}}\geq\underline{\mathbf{0}}}\frac{n_f!}{(n_f-k)!}\mathrm{E}\left(\left(U_1^{(k)}\right)^{n_1}\cdots\left(U_s^{(k)}\right)^{n_s-k}\cdots\left(U_r^{(k)}\right)^{n_r}\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!},\end{aligned}$$

which imply the desired identity. ■

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