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Recursiveness properties for multipartitional polynomials

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Abstract: In this paper, we establish recursiveness properties for multipartitional polynomials and their connection with the polynomials of multinomial type and with their derivatives. Various comprehensive examples are illustrated.

Keywords: Multipartitional polynomials; derivatives of polynomial sequences of multinomial type; recursiveness relations.

Résumé : Le but de ce papier est d'établir des propriétés de récursivité pour les polynômes multipartitionnels et leurs connexions avec les polynômes de type multinomiales et avec leurs dérivées. Divers exemples d'illustration sont présentés.

Mots clés : Polynômes multipartitionnels; polynômes de type multinomiales; relations de récursivité.

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1 Introduction

We can find an introduction and some developments of the partial and complete Bell polynomials in [1, 4, 6, 7, 8, 12, 14, 15]. An extension to the partial and complete bipartitional polynomials are developed in [3, 5, 10]. For the generalized case, the partial and complete multipartitional polynomials are considered by the authors in [2, 9, 11], see also [13]. For example, ones of their applications appear in Tutte polynomials and the inverse relations, see [9, 13]. The large application of the multipartitional polynomials gives a motivation to develop this mathematical tool. In precedent works [2, 11], the authors established identities on multipartitional polynomials. They gave some results concerning the preservation of sequences of multinomial type and some other results.
In this paper, we give new identities related to multipartitional polynomials and we establish connections with polynomials of multinomial type and with their derivatives. We also give some recursiveness properties and illustrate each result by various applications. To give some background materials, we denote:

- for $\underline{\mathbf{a}} = (a_1, \ldots, a_r) \in \mathbb{R}^r$, $\underline{\mathbf{b}} = (b_1, \ldots, b_r) \in \mathbb{R}^r$ we set

  $\underline{\mathbf{a}}.\underline{\mathbf{b}} = a_1b_1 + \cdots + a_rb_r$, $\underline{\mathbf{a}} + \underline{\mathbf{b}} = (a_1 + b_1, \ldots, a_r + b_r)$, $\lambda\underline{\mathbf{a}} = (\lambda a_1, \ldots, \lambda a_r)$,

  $(\underline{\mathbf{a}}\geq\underline{\mathbf{b}}) \Leftrightarrow (a_1 \geq b_1, \ldots, a_r \geq b_r)$, $(\underline{\mathbf{a}}>\underline{\mathbf{b}}) \Leftrightarrow (a_1 > b_1, \ldots, a_r > b_r)$,

  $\varphi(\underline{\mathbf{a}},\underline{\mathbf{b}}) = 1$ if $\underline{\mathbf{a}}\geq \underline{\mathbf{b}}$ and 0 otherwise,

- for $\underline{\mathbf{n}} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we set

  $\underline{\mathbf{t}}^{\underline{\mathbf{n}}} = t_1^{n_1} \cdots t_r^{n_r}$, $\frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!} = \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}$, $\underline{\mathbf{n}}! = n_1! \cdots n_r!$, $|\underline{\mathbf{n}}| = n_1+\cdots+n_r$, $\underline{\mathbf{1}} = (1, \ldots, 1)$,

  $\prod_{\underline{\mathbf{i}}=\underline{\mathbf{0}}}^{\underline{\mathbf{n}}} (k_{\underline{\mathbf{i}}}!) = \prod_{i_r=0}^{n_r} \cdots \prod_{i_1=0}^{n_1} (k_{i_1,\cdots,i_r}!)$, $k_{\underline{\mathbf{i}}} = k_{i_1,\cdots,i_r}$ with the convention $k_{\underline{\mathbf{0}}} = 0$,

  $\binom{\underline{\mathbf{n}}}{\underline{\mathbf{j}}} = \frac{\underline{\mathbf{n}}!}{\underline{\mathbf{j}}!(\underline{\mathbf{n}}-\underline{\mathbf{j}})!}$, $\underline{\mathbf{n}}, \underline{\mathbf{j}} \in \mathbb{N}^r$, $\binom{\underline{\mathbf{n}}}{\underline{\mathbf{j}}_1,\cdots,\underline{\mathbf{j}}_k} = \frac{\underline{\mathbf{n}}!}{\underline{\mathbf{j}}_1!\cdots\underline{\mathbf{j}}_k!}$, $\underline{\mathbf{n}}, \underline{\mathbf{j}}_1, \cdots, \underline{\mathbf{j}}_k \in \mathbb{N}^r$ with $\underline{\mathbf{j}}_1+\cdots+\underline{\mathbf{j}}_k = \underline{\mathbf{n}}$,

- for $(m, n) \in \mathbb{N}^2$ we set

  $B_{m,n;k}(x_{i,j}) = B_{m,n;k}(x_{0,1}, x_{1,0}, \cdots, x_{0,i+j}, x_{1,i+j-1}, \ldots, x_{i+j-1,1}, x_{i+j,0}, \cdots)$,

  $A_{m,n}(x_{i,j}) = A_{m,n}(x_{0,1}, x_{1,0}, \cdots, x_{0,i+j}, x_{1,i+j-1}, \ldots, x_{i+j-1,1}, x_{i+j,0}, \cdots)$,

  and for $\underline{\mathbf{n}} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we set

  $B_{\underline{\mathbf{n}},k}(x_{\underline{\mathbf{i}}}) = B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1}, \ldots, x_{1,0,\ldots,0}, x_{0,\ldots,0,2}, x_{0,\ldots,0,1,1}, \ldots)$, $\underline{\mathbf{n}} \geq \underline{\mathbf{0}}$, $k \geq 0$ and $= 0$ otherwise,

  $A_{\underline{\mathbf{n}}}(x_{\underline{\mathbf{i}}}) = A_{n_1,\ldots,n_r}(x_{0,\ldots,0,1}, \ldots, x_{1,0,\ldots,0}, x_{0,\ldots,0,2}, x_{0,\ldots,0,1,1}, \ldots)$, $\underline{\mathbf{n}} \geq \underline{\mathbf{0}}$ and $= 0$ otherwise,

  in the $B_{n_1,\ldots,n_r,k}$ and $A_{n_1,\ldots,n_r}$, the variable $x_{n_1,\ldots,n_r}$ is before $x_{m_1,\ldots,m_r}$ if $|\underline{\mathbf{n}}| < |\underline{\mathbf{m}}|$ and when $|\underline{\mathbf{n}}| = |\underline{\mathbf{m}}|$ we use the lexicographical order,

- for $\underline{\mathbf{n}} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, we set $f_{\underline{\mathbf{n}}}(x) = f_{n_1,\ldots,n_r}(x)$ if $\underline{\mathbf{n}} \geq \underline{\mathbf{0}}$, $f_{\underline{\mathbf{0}}}(x) = 1$ and $= 0$ otherwise, The complete (exponential) multipartitional polynomials $A_{\underline{\mathbf{n}}}$ in the variables $(x_{\underline{\mathbf{i}}},\ \underline{\mathbf{i}} \neq \underline{\mathbf{0}})$, can be defined by their generating function given by

$$\sum_{\underline{\mathbf{n}} \geq \underline{\mathbf{0}}} A_{\underline{\mathbf{n}}}\left(x_{\underline{\mathbf{j}}}\right) \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!} = \exp\left(\sum_{|\underline{\mathbf{i}}| \geq 1} x_{\underline{\mathbf{i}}} \frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\right), \tag{1}$$

and The complete (exponential) multipartitional polynomials $B_{\underline{\mathbf{n}};k}$ can be defined by their generating function given by

$$\sum_{|\underline{\mathbf{n}}| \geq k} B_{\underline{\mathbf{n}};k}\left(x_{\underline{\mathbf{j}}}\right) \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!} = \frac{1}{k!}\left(\sum_{|\underline{\mathbf{i}}| \geq 1} x_{\underline{\mathbf{i}}} \frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\right)^{k}. \tag{2}$$

The polynomials of multinomial type $(f_{\underline{\mathbf{n}}}(x), \underline{\mathbf{n}} \in \mathbb{N}^s)$ are defined by $f_{\underline{\mathbf{0}}}(x) := 1$ and

$$\left(\sum_{\underline{\mathbf{i}} \geq \underline{\mathbf{0}}} f_{\underline{\mathbf{i}}}(1) \frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\right)^{x} = \sum_{\underline{\mathbf{n}} \geq \underline{\mathbf{0}}} f_{\underline{\mathbf{n}}}(x) \frac{\underline{\mathbf{t}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}. \tag{3}$$

From the above definitions, one can verify that, for all real numbers $\alpha_1, \ldots, \alpha_r, \beta$, we have

$$A_{\underline{\mathbf{n}}}\left(\underline{\alpha}^{\underline{\mathbf{i}}} x_{\underline{\mathbf{i}}}\right) = \underline{\alpha}^{\underline{\mathbf{n}}} A_{\underline{\mathbf{n}}}\left(x_{\underline{\mathbf{i}}}\right) \quad \text{and} \quad B_{\underline{\mathbf{n}};k}\left(\beta \underline{\alpha}^{\underline{\mathbf{i}}} x_{\underline{\mathbf{i}}}\right) = \beta^k \underline{\alpha}^{\underline{\mathbf{n}}} B_{\underline{\mathbf{n}};k}\left(x_{\mathbf{i}}\right). \tag{4}$$

In the following, $\underline{\mathbf{s}}$, $\underline{\mathbf{s}}_1$ design vectors of $\mathbb{N}^s - \{\underline{\mathbf{0}}\}$ and $\underline{\mathbf{r}}$, $\underline{\mathbf{r}}_1$ design vectors of $\mathbb{N}^s$.
The material used here is the above definitions and the theorems given in [2, 11] by

Theorem 1 *[11]Let $s \geq 1$ be an integer and $\underline{\mathbf{a}} = (a_1, \ldots, a_s) \in \mathbb{R}^s$. If $(f_{\underline{\mathbf{n}}}(x), \underline{\mathbf{n}} \in \mathbb{N}^s)$ is a multinomial type sequence of polynomials, then the polynomials $(h_{\underline{\mathbf{n}}}(x))$ and $(p_{\underline{\mathbf{n}}}(x))$ given by*

$$\begin{aligned} h_{\underline{\mathbf{n}}}(x) &:= \frac{x}{\underline{\mathbf{a}}.\underline{\mathbf{n}} + x} f_{\underline{\mathbf{n}}}(\underline{\mathbf{a}}.\underline{\mathbf{n}} + x), \\ p_{\underline{\mathbf{n}}}(x) &:= f_{|\underline{\mathbf{n}}|}(x), \end{aligned}$$

are of multinomial type.

Theorem 2 *[2]Let $\alpha \in \mathbb{R}$, $\underline{\mathbf{a}} = (a_1, \ldots, a_s) \in \mathbb{R}^s$ and $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}} \in \mathbb{N}^s)$ be a multinomial type sequence of polynomials. We have*

$$B_{\underline{\mathbf{n}};k}\left(\alpha \binom{\underline{\mathbf{j}}}{\underline{\mathbf{s}}} \frac{f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}\left(\underline{\mathbf{a}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+\alpha\right)}{\underline{\mathbf{a}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+\alpha}\right) = \frac{\alpha}{(k-1)!}\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{s}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}} \frac{f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}\left(\underline{\mathbf{a}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+\alpha k\right)}{\underline{\mathbf{a}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+\alpha k}.$$

Theorem 3 *[2]Let $\alpha \in \mathbb{R}$ be a real number and $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}} \in \mathbb{N}^s)$ is of multinomial type sequence of polynomials. We have*

$$B_{\underline{\mathbf{n}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{s}}} D_{z=0}\left(e^{\alpha z} f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(x+z)\right)\right) = \frac{1}{k!}\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{s}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}} D_{z=0}^{k}\left(e^{\alpha z} f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}(kx+z)\right).$$

2 Multipartitional polynomials and multinomial polynomials

Using the multinomial type sequences to give some identities for multipartitional polynomials. The obtained results generalize those given in [7, 8] for Bell polynomials.

Theorem 4 *Let r, s be integers with $r, s \geq 1$, $\underline{\mathbf{a}} \in \mathbb{R}^r$, $\underline{\mathbf{b}} \in \mathbb{R}^s$, $c \in \mathbb{R}$, $(y_{\underline{\mathbf{m}}}; \underline{\mathbf{m}} \in \mathbb{N}^r)$ be a sequence of real numbers with $y_{\underline{\mathbf{0}}} = 0$ and $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}} \in \mathbb{N}^s)$ be a multinomial type sequence of polynomials. Then, we have*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}\frac{f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c\right)}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)y_{\underline{\mathbf{i}}}\right)$$
$$=\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}\frac{f_{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck\right)}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right).$$

Proof. Let $F(\underline{\mathbf{t}})^x := \sum_{\underline{\mathbf{j}}\geq\underline{\mathbf{0}}} f_{\underline{\mathbf{j}}}(x)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}$. Use (2), (3) and (4) to get

$$\sum_{|\underline{\mathbf{m}}|+|\underline{\mathbf{n}}|\geq k} B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(y_{\underline{\mathbf{i}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}$$
$$=\frac{1}{k!}\left(\sum_{|\underline{\mathbf{i}}|+|\underline{\mathbf{j}}|\geq 1} y_{\underline{\mathbf{i}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}\right)^k$$
$$=\frac{1}{k!}\left(\sum_{|\underline{\mathbf{i}}|\geq 1} y_{\underline{\mathbf{i}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\sum_{\underline{\mathbf{j}}\geq\underline{\mathbf{r}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)\frac{\underline{\mathbf{u}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}\right)^k$$
$$=\frac{1}{k!}\frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}\left(\sum_{|\underline{\mathbf{i}}|\geq 1} y_{\underline{\mathbf{i}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\sum_{\underline{\mathbf{j}}\geq\underline{\mathbf{0}}}f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)\frac{\underline{\mathbf{u}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}\right)^k$$
$$=\frac{1}{k!}\frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}\left(\sum_{|\underline{\mathbf{i}}|\geq 1} y_{\underline{\mathbf{i}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}F(\underline{\mathbf{u}})^{\underline{\mathbf{a}}.\underline{\mathbf{i}}+c}\right)^k$$
$$=\frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}\sum_{|\underline{\mathbf{m}}|\geq k} F(\underline{\mathbf{u}})^{\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck}B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}$$
$$=\frac{1}{(\underline{\mathbf{r}}!)^k}\sum_{|\underline{\mathbf{m}}|\geq k,\ \underline{\mathbf{n}}\geq k\underline{\mathbf{r}}} f_{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{n}}}}{(\underline{\mathbf{n}}-k\underline{\mathbf{r}})!}$$
$$=\sum_{|\underline{\mathbf{m}}|\geq k,\ \underline{\mathbf{n}}\geq k\underline{\mathbf{r}}}\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}f_{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}.$$

Then, by identification, we get

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(y_{\underline{\mathbf{i}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)\right)=\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}f_{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right).\qquad(5)$$

To complete the proof, use in (5) the first multinomial type sequence given by Theorem 1 instead $f_{\underline{\mathbf{n}}}(x)$. ■

Example 1 *For* $f_{\underline{\mathbf{n}}}(\alpha) = \alpha^{|\underline{\mathbf{n}}|}$, *we get from Theorem 4*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c\right)^{|\underline{\mathbf{j}}-\underline{\mathbf{r}}|-1}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)y_{\underline{\mathbf{i}}}\right)$$
$$=\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck\right)^{|\underline{\mathbf{n}}-k\underline{\mathbf{r}}|-1}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right),$$

and for $f_{\underline{\mathbf{n}}}(\alpha) = \underline{\mathbf{n}}!\binom{\alpha\mathbf{1}}{\underline{\mathbf{n}}}$, *we get from Theorem 4*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\frac{\underline{\mathbf{j}}!\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c}\binom{\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c\right)\mathbf{1}}{\underline{\mathbf{j}}-\underline{\mathbf{r}}}y_{\underline{\mathbf{i}}}\right)$$
$$=\frac{\underline{\mathbf{n}}!\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck}\binom{\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck\right)\mathbf{1}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right).$$

By the second multinomial type sequence of Theorem 1, Theorem 4 becomes:

Corollary 5 *Let* r,s *be integers with* $r,s\geq 1$, $\underline{\mathbf{a}}\in\mathbb{R}^r$, $\underline{\mathbf{b}}\in\mathbb{R}^s$, $c\in\mathbb{R}$, $(y_{\underline{\mathbf{m}}};\underline{\mathbf{m}}\in\mathbb{N}^r)$ *be a sequence of real numbers with* $y_{\underline{\mathbf{0}}}=0$ *and* $(f_n(x);n\in\mathbb{N})$ *be a binomial type sequence of polynomials. We have*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}\frac{f_{|\underline{\mathbf{j}}-\underline{\mathbf{r}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c\right)}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)y_{\underline{\mathbf{i}}}\right) \tag{6}$$
$$=\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}\frac{f_{|\underline{\mathbf{n}}-k\underline{\mathbf{r}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck\right)}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right).$$

Example 2 *For* $f_n(\alpha)=n!\binom{\alpha}{n}$, *we get from (6)*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}\frac{\left(|\underline{\mathbf{j}}-\underline{\mathbf{r}}|\right)!\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c\right)}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c}\binom{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c}{|\underline{\mathbf{j}}-\underline{\mathbf{r}}|}y_{\underline{\mathbf{i}}}\right)$$
$$=\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}\frac{\left(|\underline{\mathbf{n}}-k\underline{\mathbf{r}}|\right)!\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck\right)}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck}\binom{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck}{|\underline{\mathbf{n}}-k\underline{\mathbf{r}}|}B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right).$$

Corollary 6 *Let* (y_n) *be a sequence of real numbers with* $y_0=0$ *and* $(f_n(x))$ *be a binomial type sequence of polynomials. Then, for* $r=s=1$, $(u,v)\in\mathbb{N}^2$, $(a,b,c)\in\mathbb{R}^3$ *in Theorem 4, we get*

$$B_{m,n;k}\left(\binom{j}{v}\frac{ai+c}{ai+b(j-v)+c}f_{j-v}\left(ai+b(j-v)+c\right)y_i\right) \tag{7}$$
$$=\binom{n}{n-kv,v,\ldots,v}\frac{am+ck}{am+b(n-kv)+ck}f_{n-kv}\left(am+b(n-kv)+ck\right)B_{m;k}\left(y_i\right),$$

and by symmetry, we get

$$B_{m,n;k}\left(\binom{i}{u}\frac{bj+c}{a(i-u)+bj+c}f_{i-u}\left(a(i-u)+bj+c\right)y_j\right) \tag{8}$$
$$=\binom{m}{m-ku,u,\ldots,u}\frac{bn+ck}{a(m-ku)+bn+ck}f_{m-ku}\left(a(m-ku)+bn+ck\right)B_{n;k}\left(y_j\right).$$

Example 3 *For $f_n(\alpha)=\alpha^n$, we get from (7) and (8)*

$$B_{m,n;k}\left(\binom{j}{v}\left(ai+b(j-v)+c\right)^{j-v-1}(ai+c)\,y_i\right)$$
$$=\binom{n}{n-kv,v,\ldots,v}\left(am+b(n-kv)+ck\right)^{n-kv-1}(am+ck)\,B_{m;k}\left(y_i\right),$$

and

$$B_{m,n;k}\left(\binom{i}{u}\left(a(i-u)+bj+c\right)^{i-u-1}(bj+c)\,y_j\right)$$
$$=\binom{m}{m-ku,u,\ldots,u}\left(a(m-ku)+bn+ck\right)^{m-ku-1}(bn+ck)\,B_{n;k}\left(y_j\right).$$

Theorem 7 *Let r,s be integers with $r,s\geq 1$, $\underline{\mathbf{a}}\in\mathbb{R}^r$, $\underline{\mathbf{b}}\in\mathbb{R}^s$, $\left(y_{\underline{\mathbf{m}}};\underline{\mathbf{m}}\in\mathbb{N}^r\right)$ be a sequence of real numbers with $y_{\underline{\mathbf{0}}}=0$ and $\left(f_{\underline{\mathbf{n}}}(x);\underline{\mathbf{n}}\in\mathbb{N}^s\right)$ be a sequence of polynomials of multinomial type. We have*

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\frac{\underline{\mathbf{a}}.\underline{\mathbf{i}}}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}}y_{\underline{\mathbf{i}}}f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}\right)\right)=\frac{\underline{\mathbf{a}}.\underline{\mathbf{m}}}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}}A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)f_{\underline{\mathbf{n}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}\right). \tag{9}$$

Proof. Set $c=0$ and $\underline{\mathbf{r}}=\mathbf{0}$ in Theorem 4 and sum the two hand sides over all possible values of k. ∎

Example 4 *For $f_{\underline{\mathbf{n}}}(\alpha)=\alpha^{|\underline{\mathbf{n}}|}$ we get from (9)*

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}\right)y_{\underline{\mathbf{i}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}\right)^{|\underline{\mathbf{j}}|-1}\right)=\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}\right)A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}\right)^{|\underline{\mathbf{n}}|-1},$$

and for $f_{\underline{\mathbf{n}}}(\alpha)=\underline{\mathbf{n}}!\binom{\alpha\mathbf{1}}{\underline{\mathbf{n}}}$, we get from (9)

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\frac{\underline{\mathbf{j}}!\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}\right)}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}}y_{\underline{\mathbf{i}}}\binom{\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}\right)\underline{\mathbf{1}}}{\underline{\mathbf{j}}}\right)=\frac{\underline{\mathbf{n}}!\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}\right)}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}}A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)\binom{\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}\right)\underline{\mathbf{1}}}{\underline{\mathbf{n}}}.$$

By (9), Theorem 7 becomes:

Corollary 8 *Let r, s be integers with $r, s \geq 1$, $\underline{\mathbf{n}} \in \mathbb{N}^s$, $\underline{\mathbf{a}} \in \mathbb{R}^r$, $\underline{\mathbf{b}} \in \mathbb{R}^s$, $(y_{\underline{\mathbf{m}}}; \underline{\mathbf{m}} \in \mathbb{N}^r)$ be a sequence of real numbers with $y_{\underline{\mathbf{0}}} = 0$ and $(f_n(x))$ be a binomial type sequence of polynomials. We have*

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\frac{\underline{\mathbf{a}}.\underline{\mathbf{i}}}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}} y_{\underline{\mathbf{i}}} f_{|\underline{\mathbf{j}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}\right)\right) = \frac{\underline{\mathbf{a}}.\underline{\mathbf{m}}}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}} A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right) f_{|\underline{\mathbf{n}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}\right). \qquad (10)$$

Example 5 *For $f_n(\alpha) = B_n(\alpha)$ we get from (10)*

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\frac{\underline{\mathbf{a}}.\underline{\mathbf{i}}}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}} y_{\underline{\mathbf{i}}} B_{|\underline{\mathbf{j}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}\right)\right) = \frac{\underline{\mathbf{a}}.\underline{\mathbf{m}}}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}} A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right) B_{|\underline{\mathbf{n}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}\right),$$

and for $f_n(\alpha) = n!\binom{\alpha}{n}$, we get from (6)

$$A_{\mathbf{m},\mathbf{n}}\left((|\underline{\mathbf{j}}|)!\frac{\underline{\mathbf{a}}.\underline{\mathbf{i}}}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}}\binom{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}}{|\underline{\mathbf{j}}|} y_{\underline{\mathbf{i}}}\right) = (|\underline{\mathbf{n}}|)!\frac{\underline{\mathbf{a}}.\underline{\mathbf{m}}}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}}\binom{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}}{|\underline{\mathbf{n}}|} A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right).$$

Corollary 9 *Let (y_n) be a sequence of real numbers with $y_0 = 0$ and $(f_n(x))$ be a binomial type sequence of polynomials. Then, for $r = s = 1$, $(m, n) \in \mathbb{N}^2$, $(a, b, c) \in \mathbb{R}^3$ in Theorem 11, we get*

$$A_{m,n}\left(\frac{ai}{ai+bj} y_i f_j(ai+bj)\right) = \frac{am}{am+bn} A_m(y_i) f_n(am+bn), \qquad (11)$$

and by symmetry, we also have

$$A_{m,n}\left(\frac{bj}{ai+bj} y_j f_i(ai+bj)\right) = \frac{bn}{am+bn} A_n(y_j) f_m(am+bn). \qquad (12)$$

3 Recursiveness in multipartitional polynomials

To give such formulae, we express any multinomial type sequence in the above identities by the multipartitional polynomials. The obtained results generalize those given in [7, 8] for Bell polynomials.

Lemma 10 *Let $(x_{\underline{\mathbf{n}}}; \underline{\mathbf{n}} \in \mathbb{N}^s)$ be a sequence of real numbers such that $x_{\underline{\mathbf{s}}} = 1$. Then, there exists a multinomial type sequence $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}} \in \mathbb{N}^s)$ such that*

$$f_{\underline{\mathbf{n}}}(k) = k!\binom{\underline{\mathbf{n}}+k\underline{\mathbf{s}}}{\underline{\mathbf{n}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}^{-1} B_{\underline{\mathbf{n}}+k\underline{\mathbf{s}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}} x_{\underline{\mathbf{j}}}\right), \quad k \in \mathbb{N}. \qquad (13)$$

Proof. Let $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}} \in \mathbb{N}^s)$ be a multinomial type sequence of polynomials with $f_{\underline{\mathbf{n}}}(1) := x_{\underline{\mathbf{n}}+\underline{\mathbf{s}}}$. Then, for $\underline{\mathbf{a}} = \underline{\mathbf{0}}$ and $\alpha = 1$ in Theorem 2, we obtain (13). ■

Theorem 11 *Let $(x_{\underline{\mathbf{n}}};\underline{\mathbf{n}}\in\mathbb{N}^s)$ and $(y_{\underline{\mathbf{m}}};\underline{\mathbf{m}}\in\mathbb{N}^r)$ be sequences of real numbers satisfying $x_{\underline{\mathbf{s}}}=1$ and $y_{\underline{\mathbf{0}}}=0$. Then, for $\underline{\mathbf{p}}\in\mathbb{N}^r$, $\underline{\mathbf{q}}\in\mathbb{N}^s$, $d\in\mathbb{N}-\{0\}$, we have*

$$\begin{aligned}&B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\frac{(t-1)!\underline{\mathbf{j}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+d\right)}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1\right)!}B_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1;t}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right)y_{\underline{\mathbf{i}}}\right)\\ =\;&\varphi\left(\underline{\mathbf{n}},k\underline{\mathbf{s}}\right)\frac{(T-1)!\underline{\mathbf{n}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+dk\right)}{\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1\right)!}B_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1;T}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right),\end{aligned}\tag{14}$$

where $t=\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+d$ and $T=\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+dk$.

Proof. Let $(f_{\underline{\mathbf{n}}}(\alpha);\underline{\mathbf{n}}\in\mathbb{N}^s)$ be multinomial type sequence defined as in (13) by

$$f_{\underline{\mathbf{n}}}(k)=k!\binom{\underline{\mathbf{n}}+k\underline{\mathbf{s}}_1}{\underline{\mathbf{n}},\underline{\mathbf{s}}_1,\ldots,\underline{\mathbf{s}}_1}^{-1}B_{\underline{\mathbf{n}}+k\underline{\mathbf{s}}_1;k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}}\right).$$

Then

$$f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(t)=t!\frac{B_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1;t}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right)}{\binom{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1}{\underline{\mathbf{j}}-\underline{\mathbf{s}},\underline{\mathbf{s}}_1,\ldots,\underline{\mathbf{s}}_1}}\quad\text{and}\quad f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}(T)=T!\frac{B_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1;T}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right)}{\binom{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1}{\underline{\mathbf{n}}-k\underline{\mathbf{s}},\underline{\mathbf{s}}_1,\ldots,\underline{\mathbf{s}}_1}}.$$

Theorem 4 implies

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{s}}}\frac{f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(t)}{t}\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+d\right)y_{\underline{\mathbf{i}}}\right)=\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{s}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}\frac{f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}(T)}{T}\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+dk\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right).\tag{15}$$

To obtain (14), replace in (15) $f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(t)$ and $f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}(T)$ by their expressions given above. ■

Corollary 12 *Let $p,q\in\mathbb{N}$; $u,v,d\in\mathbb{N}-\{0\}$; $(x_n;n\in\mathbb{N})$ and $(y_n;n\in\mathbb{N})$ be sequences of real numbers satisfying $x_u=1$ and $y_0=0$. Then, we have*

$$\begin{aligned}&B_{m,n;k}\left(\varphi(j,v)(t-1)!j!\frac{B_{j-v+tu;\ t}\left(\binom{j'}{v}x_{j'}\right)}{(j-v+tu)!}(pi+d)\,y_i\right)\\ =\;&\varphi(n,kv)(T-1)!n!\frac{B_{n-kv+Tu;\ T}\left(\binom{j'}{v}x_{j'}\right)}{(n-kv+Tu)!}(pm+dk)\,B_{m;k}(y_i),\end{aligned}$$

and

$$\begin{aligned}&B_{m,n;k}\left(\varphi(i,u)(t_2-1)!i!\frac{B_{i-u+t_2v;\ t_2}\left(\binom{j'}{v}x_{j'}\right)}{(i-u+t_2v)!}(qj+d)\,y_j\right)\\ =\;&\varphi(m,ku)(T_2-1)!m!\frac{B_{m-ku+T_2v;\ T2}\left(\binom{j'}{v}x_{j'}\right)}{(m-ku+T_2v)!}(qn+dk)\,B_{n;k}(y_j),\end{aligned}$$

where $t_1=pi+q(j-v)+d$, $T_1=pm+q(n-kv)+dk$, $t_2=p(i-u)+qj+d$ and $T_2=p(m-ku)+qn+dk$.

Theorem 13 *Let* $(x_{\underline{\mathbf{n}}}; \underline{\mathbf{n}} \in \mathbb{N}^s)$ *be a sequence of real numbers such that* $x_{\underline{\mathbf{s}}} = 1$. *Then, for* $\underline{\mathbf{q}} \in \mathbb{N}^s$, $d \in \mathbb{N}-\{0\}$, *we have*

$$B_{\underline{\mathbf{n}},k}\left(\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\frac{(t-1)!\underline{\mathbf{j}}!d}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1\right)!}B_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1;t}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right)\right)$$
$$=\varphi\left(\underline{\mathbf{n}},k\underline{\mathbf{s}}\right)\frac{(T-1)!\underline{\mathbf{n}}!d}{\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1\right)!\,(k-1)!}B_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1;T}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right),$$

where $t = \underline{\mathbf{q}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+d$ *and* $T = \underline{\mathbf{q}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+dk$.

Proof. Let $(f_{\underline{\mathbf{n}}}(\alpha); \underline{\mathbf{n}} \in \mathbb{N}^s)$ be multinomial type sequence defined as in (13) by

$$f_{\underline{\mathbf{n}}}(k) = k!\binom{\underline{\mathbf{n}}+k\underline{\mathbf{s}}_1}{\underline{\mathbf{n}},\underline{\mathbf{s}}_1,\ldots,\underline{\mathbf{s}}_1}^{-1}B_{\underline{\mathbf{n}}+k\underline{\mathbf{s}}_1;k}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right), \quad k \in \mathbb{N}. \tag{16}$$

Theorem 2 becomes

$$B_{\underline{\mathbf{n}};k}\left(d\binom{\underline{\mathbf{j}}}{\underline{\mathbf{s}}}\frac{f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}\left(\underline{\mathbf{q}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+d\right)}{\underline{\mathbf{q}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+d}\right) = \frac{d}{(k-1)!}\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{s}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}\frac{f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}\left(\underline{\mathbf{q}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+dk\right)}{\underline{\mathbf{q}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+dk}. \tag{17}$$

However by (16), we have

$$\begin{aligned} f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(t) &= t!\binom{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1}{\underline{\mathbf{j}}-\underline{\mathbf{s}},\underline{\mathbf{s}}_1,\ldots,\underline{\mathbf{s}}_1}^{-1}B_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1;t}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right), \\ f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}(T) &= T!\binom{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1}{\underline{\mathbf{n}}-k\underline{\mathbf{s}},\underline{\mathbf{s}}_1,\ldots,\underline{\mathbf{s}}_1}^{-1}B_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1;T}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right). \end{aligned}$$

To obtain the desired identity, replace in (17) $f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(t)$ and $f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}}(T)$ by their expressions given above. ■

Corollary 14 *Let* $q \in \mathbb{N}$; $u, v, d \in \mathbb{N}-\{0\}$ *and* $(x_n; n \in \mathbb{N})$ *be a sequence of real numbers such that* $x_u = 1$. *Then, we have*

$$B_{n;k}\left(\varphi(j,v)\left(q(j-v)+d-1\right)!j!d\frac{B_{(qu+1)(j-v)+du;\ q(j-v)+d}\left(\binom{j'}{v}x_{j'}\right)}{\left((qu+1)(j-v)+du\right)!}\right)$$
$$= \varphi(n,kv)\left(q(n-kv)+dk-1\right)!n!\frac{d}{(k-1)!}\frac{B_{(qu+1)(n-kv)+dku;\ q(n-kv)+dk}\left(\binom{j'}{v}x_{j'}\right)}{\left((qu+1)(n-kv)+dku\right)!}.$$

Remark 1 *For the choice* $\underline{\mathbf{s}}_1 = \underline{\mathbf{s}}$ *in Theorem 13, we obtain Theorem 9 of [2]. For* $u = v = 1$ *in Corollary 14, we obtain Proposition 4 of [7].*

Theorem 15 *Let r, s be integers with $r, s \geq 1$; $\underline{\mathbf{p}} \in \mathbb{N}^r$, $\underline{\mathbf{q}} \in \mathbb{N}^s$ and $(y_{\underline{\mathbf{n}}})$ be a sequence of real numbers with $y_{\underline{\mathbf{0}}} = 0$. We have*

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}-1\right)!\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}\right)y_{\underline{\mathbf{i}}}\frac{B_{\underline{\mathbf{j}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)\underline{\mathbf{s}};\ \underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right)}{\binom{\underline{\mathbf{j}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)\underline{\mathbf{s}}}{\underline{\mathbf{j}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}}\right)$$
$$= \left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}-1\right)!\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}\right)A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)\frac{B_{\underline{\mathbf{n}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)\underline{\mathbf{s}};\ \underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right)}{\binom{\underline{\mathbf{n}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)\underline{\mathbf{s}}}{\underline{\mathbf{n}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}}.$$

Proof. Let $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}} \in \mathbb{N}^s)$ be multinomial type sequence defined as in (13). Identity (9) implies

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\frac{\underline{\mathbf{p}}.\underline{\mathbf{i}}}{\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}}y_{\underline{\mathbf{i}}}f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)\right) = \frac{\underline{\mathbf{p}}.\underline{\mathbf{m}}}{\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}}A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)f_{\underline{\mathbf{n}}}\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right). \tag{18}$$

However by (13), we have

$$f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right) = \frac{\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)!}{\binom{\underline{\mathbf{j}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)\underline{\mathbf{s}}}{\underline{\mathbf{j}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}}B_{\underline{\mathbf{j}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)\underline{\mathbf{s}};\ \underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right),$$
$$f_{\underline{\mathbf{n}}}\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right) = \frac{\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)!}{\binom{\underline{\mathbf{n}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)\underline{\mathbf{s}}}{\underline{\mathbf{n}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}}B_{\underline{\mathbf{n}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)\underline{\mathbf{s}};\ \underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right).$$

To obtain the desired identity, replace in (18) $f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)$ and $f_{\underline{\mathbf{n}}}\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)$ by their expressions given above. ∎

4 Multipartitional polynomials and derivatives of multinomial polynomials

In this section, we use Theorem 3 and Theorems of the last section to deduce many applications related to the recursiveness of multipartitional polynomials. We use the convention $D_{z=0}^{-1}g(z) = 0$.

Theorem 16 *Let r, s be integers with $r, s \geq 1$, $\underline{\mathbf{a}} \in \mathbb{R}^r$, $\underline{\mathbf{b}} \in \mathbb{R}^s$, $c \in \mathbb{R}$, $(y_{\underline{\mathbf{m}}}; \underline{\mathbf{m}} \in \mathbb{N}^r)$ be a sequence of real numbers with $y_{\underline{\mathbf{0}}} = 0$ and $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}} \in \mathbb{N}^s)$ be a multinomial type sequence of polynomials. We have*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}D_{z=0}\left[\frac{\underline{\mathbf{a}}.\underline{\mathbf{i}}+c+z}{t+z}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}(t+z)e^{\alpha z}\right]y_{\underline{\mathbf{i}}}\right)$$
$$= \binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}D_{z=0}^{k}\left[\frac{\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck+z}{T+z}f_{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}(T+z)e^{\alpha z}\right]B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right),$$

where $t = \underline{\mathbf{a}}.\underline{\mathbf{i}} + \underline{\mathbf{b}}.\left(\underline{\mathbf{j}} - \underline{\mathbf{r}}\right) + c$ and $T = \underline{\mathbf{a}}.\underline{\mathbf{m}} + \underline{\mathbf{b}}.\left(\underline{\mathbf{n}} - k\underline{\mathbf{r}}\right) + ck$.

Proof. Let $F(\underline{\mathbf{t}})^x := \sum_{\underline{\mathbf{j}}\geq\underline{\mathbf{0}}} f_{\underline{\mathbf{j}}}(x)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}$, we have

$$\begin{aligned}
&\sum_{|\underline{\mathbf{m}}|+|\underline{\mathbf{n}}|\geq k} B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(y_{\underline{\mathbf{i}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}D_{z=0}\left[e^{\alpha z}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c+z\right)\right]\right)\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}\\
=\;& \frac{1}{k!}\left(\sum_{|\underline{\mathbf{i}}|+|\underline{\mathbf{j}}|\geq 1} y_{\underline{\mathbf{i}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}D_{z=0}\left[e^{\alpha z}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c+z\right)\right]\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}\right)^k\\
=\;& \frac{1}{k!}\left(\sum_{|\underline{\mathbf{i}}|\geq 1} y_{\underline{\mathbf{i}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\sum_{\underline{\mathbf{j}}\geq\underline{\mathbf{r}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}D_{z=0}\left[e^{\alpha z}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c+z\right)\right]\frac{\underline{\mathbf{u}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}\right)^k\\
=\;& \frac{1}{k!}\frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}\left(D_{z=0}\left(e^{\alpha z}\sum_{|\underline{\mathbf{i}}|\geq 1} y_{\underline{\mathbf{i}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}\sum_{\underline{\mathbf{j}}\geq\underline{\mathbf{0}}} f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c+z\right)\frac{\underline{\mathbf{u}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}\right)\right)^k\\
=\;& \frac{1}{k!}\frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}\left(D_{z=0}\left(e^{\alpha z}F(\underline{\mathbf{u}})^{c+z}\right)\sum_{|\underline{\mathbf{i}}|\geq 1} y_{\underline{\mathbf{i}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}F(\underline{\mathbf{u}})^{\underline{\mathbf{a}}.\underline{\mathbf{i}}}\right)^k\\
=\;& \frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}\frac{(\alpha+\ln F(\underline{\mathbf{u}}))^k F(\underline{\mathbf{u}})^{ck}}{k!}\left(\sum_{|\underline{\mathbf{i}}|\geq 1} y_{\underline{\mathbf{i}}}\frac{\underline{\mathbf{t}}^{\underline{\mathbf{i}}}}{\underline{\mathbf{i}}!}F(\underline{\mathbf{u}})^{\underline{\mathbf{a}}.\underline{\mathbf{i}}}\right)^k\\
=\;& \frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}(\alpha+\ln F(\underline{\mathbf{u}}))^k F(\underline{\mathbf{u}})^{ck}\sum_{|\underline{\mathbf{m}}|\geq k} F(\underline{\mathbf{u}})^{\underline{\mathbf{a}}.\underline{\mathbf{m}}}B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}})\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\\
=\;& \frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}D_{z=0}^k\left(e^{\alpha z}\sum_{|\underline{\mathbf{m}}|\geq k} F(\underline{\mathbf{u}})^{\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck+z}B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}})\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\right)\\
=\;& \frac{\underline{\mathbf{u}}^{k\underline{\mathbf{r}}}}{(\underline{\mathbf{r}}!)^k}\sum_{|\underline{\mathbf{m}}|\geq k,\ |\underline{\mathbf{j}}|\geq 0} D_{z=0}^k\left[e^{\alpha z}f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck+z\right)\right]B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}})\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{j}}}}{\underline{\mathbf{j}}!}\\
=\;& \sum_{|\underline{\mathbf{m}}|\geq k,\ \underline{\mathbf{n}}\geq k\underline{\mathbf{r}}}\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}D_{z=0}^k\left[e^{\alpha z}f_{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck+z\right)\right]B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}})\frac{\underline{\mathbf{t}}^{\underline{\mathbf{m}}}}{\underline{\mathbf{m}}!}\frac{\underline{\mathbf{u}}^{\underline{\mathbf{n}}}}{\underline{\mathbf{n}}!}.
\end{aligned}$$

Then

$$\begin{aligned}
&B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(y_{\underline{\mathbf{i}}}\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}D_{z=0}\left[e^{\alpha z}f_{\underline{\mathbf{j}}-\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+c+z\right)\right]\right)\\
=\;& \binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}D_{z=0}^k\left[e^{\alpha z}f_{\underline{\mathbf{n}}-k\underline{\mathbf{r}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck+z\right)\right]B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}}).
\end{aligned}$$

To obtain the desired identity, use the first multinomial type sequence of Theorem 1 instead of $(f_{\underline{\mathbf{n}}}(x))$. ■

By the second multinomial type sequence of Theorem 1, Theorem 16 becomes:

Corollary 17 *Let r,s be integers with $r,s\geq 1$, $\underline{\mathbf{n}}\in\mathbb{N}^s$, $\underline{\mathbf{a}}\in\mathbb{R}^r$, $\underline{\mathbf{b}}\in\mathbb{R}^s$, $c\in\mathbb{R}$, $(y_{\underline{\mathbf{m}}};\underline{\mathbf{m}}\in\mathbb{N}^r)$ be a sequence of real numbers with $y_{\underline{\mathbf{0}}}=0$ and $(f_n(x))$ be a binomial type* *sequence of polynomials. We have*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}},k}\left(\binom{\underline{\mathbf{j}}}{\underline{\mathbf{r}}}D_{z=0}\left[\frac{\underline{\mathbf{a}}.\underline{\mathbf{i}}+c+z}{t+z}f_{|\underline{\mathbf{j}}-\underline{\mathbf{r}}|}(t+z)e^{\alpha z}\right]y_{\underline{\mathbf{i}}}\right) \tag{19}$$
$$=\binom{\underline{\mathbf{n}}}{\underline{\mathbf{n}}-k\underline{\mathbf{r}},\underline{\mathbf{r}},\ldots,\underline{\mathbf{r}}}D_{z=0}^{k}\left[\frac{\underline{\mathbf{a}}.\underline{\mathbf{m}}+ck+z}{T+z}f_{|\underline{\mathbf{n}}-k\underline{\mathbf{r}}|}(T+z)e^{\alpha z}\right]B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}}),$$

where $t=\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{r}}\right)+c$ *and* $T=\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{r}}\right)+ck$.

Corollary 18 *Let* (y_n) *be a sequence of real numbers with* $y_0=0$ *and* $(f_n(x))$ *be a binomial type sequence of polynomials. Then, for* $r=s=1$, u, $v\in\mathbb{N}$, $(a,b,c)\in\mathbb{R}^3$ *in Theorem 16, we get*

$$B_{m,n;k}\left(\binom{j}{v}D_{z=0}\left[(ai+c+z)\frac{f_{j-v}(ai+b(j-v)+c+z)}{ai+b(j-v)+c+z}e^{\alpha z}\right]y_i\right)=\binom{n}{n-kv,v,\ldots,v}$$
$$\times D_{z=0}^{k}\left[(am+ck+z)\frac{f_{n-kv}(am+b(n-kv)+ck+z)}{am+b(n-kv)+ck+z}e^{\alpha z}\right]B_{m;k}(y_i),$$

and by symmetry, we have also

$$B_{m,n;k}\left(\binom{i}{u}D_{z=0}\left[(bj+c+z)\frac{f_{i-u}(a(i-u)+bj+c+z)}{a(i-u)+bj+c+z}e^{\alpha z}\right]y_j\right)=\binom{m}{m-ku,u,\ldots,u}$$
$$\times D_{z=0}^{k}\left[(bn+ck+z)\frac{f_{m-ku}(a(m-ku)+bn+ck+z)}{a(m-ku)+bn+ck+z}e^{\alpha z}\right]B_{n;k}(y_j).$$

Theorem 19 *Let* $(y_{\underline{\mathbf{m}}};\underline{\mathbf{m}}\in\mathbb{N}^r)$ *be a sequence of real numbers such that* $y_{\underline{\mathbf{0}}}=0$ *and* $(f_{\underline{\mathbf{n}}}(x);\underline{\mathbf{n}}\in\mathbb{N}^s)$ *be a multinomial type sequence of polynomials. Then, for* $\underline{\mathbf{a}}\in\mathbb{R}^s$, $\underline{\mathbf{p}}\in\mathbb{N}^r$, $\underline{\mathbf{q}}\in\mathbb{N}^s$, $d\in\mathbb{N}-\{0\}$, *we have*

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\frac{\underline{\mathbf{j}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+d\right)y_{\underline{\mathbf{i}}}}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}\right)!}D^{t-1}(xD+1)\left(H\left(\underline{\mathbf{j}},1,t,\underline{\mathbf{i}};z\right)e^{\alpha z}\right)\Big|_{z=0}\right) \tag{20}$$
$$=\varphi(\underline{\mathbf{n}},k\underline{\mathbf{s}})\frac{\underline{\mathbf{n}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+dk\right)B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}})}{(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}})!}D^{T-1}(xD+1)\left(H(\underline{\mathbf{n}},k,T,\underline{\mathbf{m}};z)e^{\alpha z}\right)|_{z=0},$$

where $t=\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+d$, $T=\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.(\underline{\mathbf{n}}-k\underline{\mathbf{s}})+dk$, $\underline{\mathbf{b}}=(\underline{\mathbf{a}}.\underline{\mathbf{r}}+x)\underline{\mathbf{p}}$, $\underline{\mathbf{c}}=(\underline{\mathbf{a}}.\underline{\mathbf{r}}+x)\underline{\mathbf{q}}+\underline{\mathbf{a}}$, $c=(\underline{\mathbf{a}}.\underline{\mathbf{r}}+x)d$ *and*

$$H(\underline{\mathbf{n}},k,T,\underline{\mathbf{m}};z)=\frac{f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}}\left(\underline{\mathbf{b}}.\underline{\mathbf{m}}+\underline{\mathbf{c}}.(\underline{\mathbf{n}}-k\underline{\mathbf{s}})+ck+z\right)}{\underline{\mathbf{b}}.\underline{\mathbf{m}}+\underline{\mathbf{c}}.(\underline{\mathbf{n}}-k\underline{\mathbf{s}})+ck+z}.$$

Proof. From Theorem 11, we have

$$B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\frac{(t-1)!\underline{\mathbf{j}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+d\right)}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1\right)!}B_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1;t}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right)y_{\underline{\mathbf{i}}}\right)$$
$$=\varphi(\underline{\mathbf{n}},k\underline{\mathbf{s}})\frac{(T-1)!\underline{\mathbf{n}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+dk\right)}{(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1)!}B_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1;T}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right)B_{\underline{\mathbf{m}};k}(y_{\underline{\mathbf{i}}}).$$

Then, for the choice $\underline{\mathbf{r}} = \underline{\mathbf{s}}_1 - \underline{\mathbf{s}} \geq \mathbf{0}$, $x_{\underline{\mathbf{n}}} = D_{z=0}\left(e^{\alpha z} f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(x+z)\right)$ and by Theorem 3, we obtain

$$\begin{aligned} & B_{\underline{\mathbf{m}},\underline{\mathbf{n}};k}\left(\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\frac{\underline{\mathbf{j}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+d\right)}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}\right)!t}D^{t}_{z=0}\left(e^{\alpha z} f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}}(tx+z)\right)y_{\underline{\mathbf{i}}}\right) \\ = \; & \varphi\left(\underline{\mathbf{n}},k\underline{\mathbf{s}}\right)\frac{\underline{\mathbf{n}}!\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+dk\right)}{\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}\right)!T}D^{T}_{z=0}\left(e^{\alpha z} f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}}(Tx+z)\right)B_{\underline{\mathbf{m}};k}\left(y_{\underline{\mathbf{i}}}\right). \end{aligned}$$

Use the multinomial type sequence the first multinomial type sequence of Theorem 1 instead of $\left(f_{\underline{\mathbf{n}}}(x);\underline{\mathbf{n}}\in\mathbb{N}^s\right)$. ■

Theorem 20 *Let* $\left(x_{\underline{\mathbf{n}}};\underline{\mathbf{n}}\in\mathbb{N}^s\right)$ *be a sequence of real numbers such that* $x_{\underline{\mathbf{s}}} = 1$ *and* $\left(f_{\underline{\mathbf{n}}}(x);\underline{\mathbf{n}}\in\mathbb{N}^s\right)$ *be a multinomial type sequence of polynomials. Then, for* $\underline{\mathbf{a}} \in \mathbb{R}^s$, $\underline{\mathbf{q}} \in \mathbb{N}^s$, $d \in \mathbb{N}-\{0\}$, *we have*

$$\begin{aligned} & B_{\underline{\mathbf{n}},k}\left(\frac{\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\underline{\mathbf{j}}!d}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}\right)!}D^{t-1}(xD+1)\left(\frac{f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}}\left(\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+c+z\right)}{\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+c+z}e^{\alpha z}\right)\Bigg|_{z=0}\right) \qquad (21) \\ = \; & \frac{\varphi\left(\underline{\mathbf{n}},k\underline{\mathbf{s}}\right)\underline{\mathbf{n}}!d}{\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}\right)!(k-1)!}D^{T-1}(xD+1)\left(\frac{f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}}\left(\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+ck+z\right)}{\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+ck}e^{\alpha z}\right)\Bigg|_{z=0}, \end{aligned}$$

where $t = \underline{\mathbf{q}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+d$, $T = \underline{\mathbf{q}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+dk$, $\underline{\mathbf{b}} = \underline{\mathbf{a}} + \left(\underline{\mathbf{a}}.\underline{\mathbf{r}}+x\right)\underline{\mathbf{q}}$ *and* $c = \left(\underline{\mathbf{a}}.\underline{\mathbf{r}}+x\right)d$.

Proof. From Theorem 13, we have

$$\begin{aligned} & B_{\underline{\mathbf{n}},k}\left(\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\frac{(t-1)!\underline{\mathbf{j}}!d}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1\right)!}B_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{s}}_1;t}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right)\right) \\ = \; & \varphi\left(\underline{\mathbf{n}},k\underline{\mathbf{s}}\right)\frac{(T-1)!\underline{\mathbf{n}}!d}{\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1\right)!(k-1)!}B_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{s}}_1;T}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}_1}x_{\underline{\mathbf{j}}'}\right). \end{aligned}$$

For the choice $x_{\underline{\mathbf{n}}} = D_{z=0}\left(e^{\alpha z} f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}_1}(x+z)\right)$ and by Theorem 3, we obtain

$$\begin{aligned} & B_{\underline{\mathbf{n}},k}\left(\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\frac{\underline{\mathbf{j}}!d}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}\right)!t}D^{t}_{z=0}\left(e^{\alpha z} f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}}(tx+z)\right)\right) \\ = \; & \varphi\left(\underline{\mathbf{n}},k\underline{\mathbf{s}}\right)\frac{\underline{\mathbf{n}}!d}{\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}\right)!(k-1)!T}D^{T}_{z=0}\left(e^{\alpha z} f_{\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}}(Tx+z)\right), \end{aligned}$$

where $\underline{\mathbf{s}}_1$, $\underline{\mathbf{s}}$ are chosen such that $\underline{\mathbf{s}}_1 \geq \underline{\mathbf{s}}$ and $\underline{\mathbf{r}} = \underline{\mathbf{s}}_1 - \underline{\mathbf{s}}$. Use the first multinomial type sequence of Theorem 1 instead of $\left(f_{\underline{\mathbf{n}}}(x)\right)$. ■

Remark 2 *For* $\underline{\mathbf{r}} = \mathbf{0}$ *in Theorem 20, we obtain Theorem 20 of [2].*

Corollary 21 *Let* $(x_{\underline{\mathbf{n}}};\underline{\mathbf{n}}\in\mathbb{N}^s)$ *be a sequence of real numbers such that* $x_{\underline{\mathbf{s}}}=1$ *and* $(f_n(x))$ *be a binomial type sequence of polynomials. Then, for* $\underline{\mathbf{a}}\in\mathbb{R}^s$, $\underline{\mathbf{q}}\in\mathbb{N}^s$, $d\in\mathbb{N}-\{0\}$, *we have*

$$B_{\underline{\mathbf{n}},k}\left(\frac{\varphi\left(\underline{\mathbf{j}},\underline{\mathbf{s}}\right)\underline{\mathbf{j}}!d}{\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}\right)!}D^{t-1}(xD+1)\left(\frac{f_{\left|\underline{\mathbf{j}}-\underline{\mathbf{s}}+t\underline{\mathbf{r}}\right|}\left(\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+c+z\right)}{\underline{\mathbf{b}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+c+z}e^{\alpha z}\right)\Bigg|_{z=0}\right)$$
$$=\frac{\varphi\left(\underline{\mathbf{n}},k\underline{\mathbf{s}}\right)\underline{\mathbf{n}}!d}{\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}\right)!(k-1)!}D^{T-1}(xD+1)\left(\frac{f_{\left|\underline{\mathbf{n}}-k\underline{\mathbf{s}}+T\underline{\mathbf{r}}\right|}\left(\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+ck+z\right)}{\underline{\mathbf{b}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+ck}e^{\alpha z}\right)\Bigg|_{z=0},$$

where $t=\underline{\mathbf{q}}.\left(\underline{\mathbf{j}}-\underline{\mathbf{s}}\right)+d$, $T=\underline{\mathbf{q}}.\left(\underline{\mathbf{n}}-k\underline{\mathbf{s}}\right)+dk$, $\underline{\mathbf{b}}=\underline{\mathbf{a}}+\left(\underline{\mathbf{a}}.\underline{\mathbf{r}}+x\right)\underline{\mathbf{q}}$ *and* $c=\left(\underline{\mathbf{b}}.\underline{\mathbf{r}}+x\right)d$.

Corollary 22 *Let* $u\in\mathbb{N}$, $v\in\mathbb{N}-\{0\}$ *and* $(x_n;n\in\mathbb{N})$ *be a sequence of real numbers such that* $x_v=1$. *Then, for* $a\in\mathbb{R}$, $q\in\mathbb{N}$, $d\in\mathbb{N}-\{0\}$, *we have*

$$B_{n,k}\left(\varphi(j,v)\frac{j!d}{h!}D^{q(j-v)+d-1}(xD+1)\left(\frac{(f_h(b(j-v)+c+z))}{b(j-v)+c+z}e^{\alpha z}\right)\Bigg|_{z=0}\right)$$
$$=\varphi(n,kv)\frac{n!d}{H!(k-1)!}D^{q(n-kv)+dk-1}(xD+1)\left(\frac{f_H(b(n-kv)+ck+z)}{b(n-kv)+ck+z}e^{\alpha z}\right)\Bigg|_{z=0},$$

where $b=a+(au+x)q$, $c=(au+x)d$, $h=(qu+1)(j-v)+du$ *and* $H=(qu+1)(n-kv)+dku$.

Theorem 23 *Let* r,s *be integers with* $r,s\geq1$; $\underline{\mathbf{c}}\in\mathbb{R}^s$, $\underline{\mathbf{p}}\in\mathbb{N}^r$, $\underline{\mathbf{q}}\in\mathbb{N}^s$, $(y_{\underline{\mathbf{m}}};\underline{\mathbf{m}}\in\mathbb{N}^r)$ *be a sequence of real numbers with* $y_{\underline{\mathbf{0}}}=0$ *and* $\left(f_{\underline{\mathbf{n}}}(x);\underline{\mathbf{n}}\in\mathbb{N}^s\right)$ *be a multinomial type sequence of polynomials. We have*

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}\right)y_{\underline{\mathbf{i}}}D^{\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}-1}\left((xD+1)\frac{f_{\underline{\mathbf{j}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}+z\right)}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}+z}e^{\alpha z}\right)\Bigg|_{z=0}\right) \tag{22}$$
$$=\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}\right)A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)D^{\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}-1}\left((xD+1)\frac{f_{\underline{\mathbf{n}}}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}+z\right)}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}+z}e^{\alpha z}\right)\Bigg|_{z=0},$$

where $\underline{\mathbf{a}}=x\underline{\mathbf{p}}$ *and* $\underline{\mathbf{b}}=\underline{\mathbf{c}}+x\underline{\mathbf{q}}$.

Proof. From Theorem 15, we have

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}-1\right)!\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}\right)y_{\underline{\mathbf{i}}}\frac{B_{\underline{\mathbf{j}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)\underline{\mathbf{s}};\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right)}{\binom{\underline{\mathbf{j}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)\underline{\mathbf{s}}}{\underline{\mathbf{j}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}}\right)$$
$$=\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}-1\right)!\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}\right)A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)\frac{B_{\underline{\mathbf{n}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)\underline{\mathbf{s}};\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}}\left(\binom{\underline{\mathbf{j}}'}{\underline{\mathbf{s}}}x_{\underline{\mathbf{j}}'}\right)}{\binom{\underline{\mathbf{n}}+\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)\underline{\mathbf{s}}}{\underline{\mathbf{n}},\underline{\mathbf{s}},\ldots,\underline{\mathbf{s}}}}.$$

For the choice $x_{\underline{\mathbf{n}}}=D_{z=0}\left(e^{\alpha z}f_{\underline{\mathbf{j}}-\underline{\mathbf{s}}}(x+z)\right)$ by Theorem 3, we obtain

$$A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left(\frac{\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}\right)y_{\underline{\mathbf{i}}}}{\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}}D_{z=0}^{\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}}\left(e^{\alpha z}f_{\underline{\mathbf{j}}}\left(\left(\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}\right)x+z\right)\right)\right)$$
$$=\frac{\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}\right)A_{\underline{\mathbf{m}}}\left(y_{\underline{\mathbf{i}}}\right)}{\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}}D_{z=0}^{\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}}\left(e^{\alpha z}f_{\underline{\mathbf{n}}}\left(\left(\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}\right)x+z\right)\right).$$

Use the multinomial type sequence the first multinomial type sequence of Theorem 1 instead of $(f_{\underline{\mathbf{n}}}(x); \underline{\mathbf{n}}\in\mathbb{N}^s)$. ■

Corollary 24 *Let r,s be integers with $r,s \geq 1$; $\underline{\mathbf{c}} \in \mathbb{R}^s$, $\underline{\mathbf{p}} \in \mathbb{N}^r$, $\underline{\mathbf{q}} \in \mathbb{N}^s$, $(y_{\underline{\mathbf{m}}}; \underline{\mathbf{m}}\in\mathbb{N}^r)$ be a sequence of real numbers with $y_{\underline{\mathbf{0}}} = 0$ and $(f_n(x))$ be a binomial type sequence of polynomials. We have*

$$\begin{aligned} & A_{\underline{\mathbf{m}},\underline{\mathbf{n}}}\left((\underline{\mathbf{p}}.\underline{\mathbf{i}})\, y_{\underline{\mathbf{i}}}\, D^{\underline{\mathbf{p}}.\underline{\mathbf{i}}+\underline{\mathbf{q}}.\underline{\mathbf{j}}-1}\left((xD+1)\frac{f_{|\underline{\mathbf{j}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}+z\right)}{\underline{\mathbf{a}}.\underline{\mathbf{i}}+\underline{\mathbf{b}}.\underline{\mathbf{j}}+z}e^{\alpha z}\right)\Bigg|_{z=0}\right) \\ = \;& (\underline{\mathbf{p}}.\underline{\mathbf{m}})\, A_{\underline{\mathbf{m}}}(y_{\underline{\mathbf{i}}})\, D^{\underline{\mathbf{p}}.\underline{\mathbf{m}}+\underline{\mathbf{q}}.\underline{\mathbf{n}}-1}\left((xD+1)\frac{f_{|\underline{\mathbf{n}}|}\left(\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}+z\right)}{\underline{\mathbf{a}}.\underline{\mathbf{m}}+\underline{\mathbf{b}}.\underline{\mathbf{n}}+z}e^{\alpha z}\right)\Bigg|_{z=0}, \end{aligned}$$

where $\underline{\mathbf{a}} = x\underline{\mathbf{p}}$ and $\underline{\mathbf{b}} = \underline{\mathbf{a}} + x\underline{\mathbf{q}}$.

Corollary 25 *Let $c \in \mathbb{R}$, p, q, u, $v \in \mathbb{N}$, (y_n) be a sequence of real numbers with $y_0 = 0$ and $(f_n(x))$ be a binomial type sequence of polynomials. We have*

$$\begin{aligned} & A_{m,n}\left((pi)\, y_i\, D^{pi+qj-1}(xD+1)\left(\frac{f_j(ai+bj+z)}{ai+bj+z}e^{\alpha z}\right)\Bigg|_{z=0}\right) \\ = \;& (pm)\, A_m(y_i)\, D^{pm+qn-1}(xD+1)\left(\frac{f_n(am+bn+z)}{am+bn+z}e^{\alpha z}\right)\Bigg|_{z=0}, \end{aligned}$$

where $a = xp$ and $b = c + xq$; and

$$\begin{aligned} & A_{m,n}\left((pj)\, y_j\, D^{pi+qj-1}(xD+1)\left(\frac{f_i(ai+bj+z)}{ai+bj+z}e^{\alpha z}\right)\Bigg|_{z=0}\right) \\ = \;& (qn)\, A_n(y_j)\, D^{pm+qn-1}(xD+1)\left(\frac{f_m(am+bn+z)}{am+bn+z}e^{\alpha z}\right)\Bigg|_{z=0}, \end{aligned}$$

where $a = c + xp$ and $b = xq$.

References

[1] E. T. Bell, Exponential polynomials. *Ann. Math.* **35** (1934), 258-277.

[2] H. Belbachir, M. Mihoubi, The (exponential) multipartitional polynomials and polynomial sequences of multinomial type, Part II. Arab J Math Sci (2013), http://dx.doi.org/10.1016/j.ajmsc.2013.06.003.

[3] H. Belbachir, M. Mihoubi, The (exponential) bipartitional polynomials and polynomial sequences of trinomial type, Part I. Integers **11** (2011), A29, 12 pp.

[4] D. Birmajer, J. B. Gil, M. D Weiner, Some convolution identities and an inverse relation involving partial Bell polynomials, The Electronic Journal of Combinatorics, **19(4)** (2012), #P34.

[5] C. A. Charalambides, *Enumerative Combinatorics.* CHAPMAN & HALL/CRC, A CRC Press Company, Boca Raton, London, New York, Washington, D.C. 2001.

[6] L. Comtet, *Advanced Combinatorics.* Dordrecht, Netherlands: Reidel, 1974.

[7] M. Mihoubi, Bell polynomials and binomial type sequences. *Discrete Math.* **308** (2008), 2450–2459.

[8] M. Mihoubi, The role of binomial type sequences in determination identities for Bell polynomials. *Ars Combin.* 108, 323–337, (2013).

[9] M. Mihoubi, H. Belbachir, Polynomials of multipartitionnal type and inverse relations. Discussiones Mathematicae: General Algebra and Applications 31 (2011) 185–200.

[10] M. Mihoubi, H. Belbachir, The (exponential) bipartitional polynomials and polynomial sequences of trinomial type, Part II. Integers **11** (2011), A18, 17 pp.

[11] M. Mihoubi, H. Belbachir, The (exponential) multipartitional polynomials and polynomial sequences of multinomial type, Part I. Arab J. Math Sci. (2013), http://dx.doi.org/10.1016/j.ajmsc.2012.12.001.

[12] J. Pitman. *Combinatorial stochastic processes,* volume (1875) of Lecture Notes in Mathematics. Springer-Verlag, Berlin, (2006). Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July (2002).

[13] A. D. Scott and A. D. Sokal, Some variants of the exponential formula, with application to the multivariate Tutte polynomial (Alias potts model). *Séminaire Lotharingien de Combinatoire* 61A (2009), Article B61Ae.

[14] J. Riordan, *Combinatorial Identities.* Huntington, New York, 1979.

[15] S. Roman, *The Umbral Calculus.* New York: Academic Press, 1984.