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An exact method to solve the multi-objective minimum spanning tree problem

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Abstract: An exact method is described to generate the efficient set of the multi-objective minimum spanning tree problem. It is based on a branch and bound method and uses a branching process with respect to particular edges of the given graph, inducing a step of constructing constraints of spanning tree problem at each node of a search tree. This has the effect of partitioning the initial graph into sub-graphs, each of which corresponds to a discrete multi-objective linear program allowing to find the efficient set of spanning trees.

Keywords: Minimum spanning tree, integer linear programming, multiple objective linear optimization, combinatorial optimization.

Résumé : Dans cette étude, une méthode exacte est présentée pour le problème de l'arbre multi-objectif. Elle est basée sur un principe de séparation par rapport à des arêtes particulières du graphe, induisant une étape de construction des contraintes du problème, de proche en proche, dans une structure arborescente. Ceci a pour effet de partitionner le graphe initial en sous graphes, chacun correspondant à un programme multi-objectif linéaire discret permettant de trouver des arbres efficaces du problème.

Mots clés : Arbre, programme linéaire en variables entières, optimisation multi-objectif linéaire.

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1 Introduction

The problem of multi-objective minimum spanning tree (MOST) is encountered in many practical applications such as optimization of the service's quality in a telecommunication network whose evaluation is represented by a set of criteria: minimum delay, maximum flow, minimum error rate, minimum operating cost ... etc [4]. In this study, we describe an exact method to find the efficient set of MOST in an undirected connected graph in which a cost vector of dimension p≥2 is associated with each edge. This problem is known to be NP-hard even for $p = 2$ [1, 7] and although some methods developed for $p = 2$ were theoretically generalized for $p \geq 3$ [7, 11], no implementation was made. In [12], the authors proposed an algorithm to solve the bi-objective case based on two phases. The first phase calculates two supported efficient solutions obtained by solving the single objective spanning tree problem for each criterion. To generate the non-supported solutions, a â euro œk-bestâ euro ? algorithm [5] or the branch-and-bound method are used to solve a weighted sum program. The authors of reference [2] aim to find all the non-dominated solutions of the bi-objective minimum spanning tree problem by reducing it into a single-objective problem. The method is based on the weighted sum function, which uses Kruskal's algorithm [8] to obtain the first supported spanning tree T1. To build a new spanning tree T2, the edges that are not in T1 are replaced by those which must leave T1, while preserving the non-dominance property. Still in the bi-objective case, a branch and bound approach is also given in reference [11]. This is mainly based on an estimation of all points not dominated by a given sub-problem that takes advantage of the two-dimensional nature of the objective space. In fact, in this case it is easy to list the entire extreme supported points of a sub-problem. Corley's algorithm [6], is described to generate the efficient set of MOST. However, a counter example which shows that the algorithm is able to find spanning trees that are not efficient has been given by Hamacher and Ruhe [7]. In fact, Corley suggests to construct iteratively spanning trees of the graph, using Prim's algorithm [10]. At each step of the algorithm, the set of vertices contained in the already sub-tree defines a cut in the original graph. The current sub-tree is increased by sub-trees as many as non-dominated edges existing in this cut. This paper describes an exact method for finding the complete efficient set of the MOST problem with no restriction on the criteria number p. Its principle search tree is based on a two-step procedure performed alternately: we start by a separation step with respect to edges in common with at least two cycles of the given graph, yielding the step of constructing the problem constraints in the form of linear equalities and inequalities. This ensures the non-existence of cycles. Each branch of the search tree induces a multi-objective discrete linear program modeling the problem of the corresponding MOST throughout this branch.

2 Definitions and notations

Given a connected and undirected simple graph $G = (V, E)$ of order n, where each edge $e_i \in E, i = \overline{1, m}$, is valued by a cost vector $c_{ki}, k = 1..r, r \geq 2$, and $t \subset E$ is a spanning tree of G. The cost vector of the spanning tree T is given by $C_k(T) = \sum_{e_i \in T} c_{ik}; k = 1..r$. We note $C(T) = (C_k)_{k=1..r}$. We say that the vector $C(T)$ dominates another vector $C(T')$ if $C_k(T) \leq C_k(T') \forall k = 1..r$ and $C_k(T) < C_k(T')$ for at least one index k. A spanning tree

T of G is efficient if there is no spanning tree T' of G such that $C(T)$ dominates $C(T')$. The ideal point I has coordinates I_k such that:

$$I_k = C_k(T) = \min\{C_k(T) \setminus T\ spanning\ tree\ of G\}, k = 1..r$$

The mathematical model associated with MOST is written as follows:

$$x_i = \begin{cases} 1 & if \quad the \quad edge \quad e_i \in T \\ 0 & otherwise \quad i = \overline{1, m} \end{cases}$$

$$(P) \begin{cases} MinZ_1 = \sum_{i=1}^{m} c_{i1} x_i \\ MinZ_2 = \sum_{i=2}^{m} c_{i2} x_i \\ \vdots \\ MinZ_r = \sum_{i=r}^{m} c_{ir} x_i \\ \sum_{j=1}^{m} = n - 1 & (1) \\ \sum_{j \in E(S)}^{n} \leq |S| - 1, \forall S \subset V, S \neq \varnothing & (2) \\ x \in \{0, 1\}^n \end{cases}$$

where $|E_S|$ is the number of edges of the sub-graph induced by the set $S, S \subset V$.

In the model (P), the number of constraints of type (2) is exponential and any attempt to solve it with an exact method is already useless in the single objective case.

3 Principle of the method

We first start by recalling an important property in the graphs, namely an edge of a graph $G = (V, E)$ is either an isthmus, or it belongs to a cycle of G. Our method is based on this property in the sense that obtaining all the spanning trees of G is done by dissecting the edges of cycles of G, particularly those common to two or more cycles. We denote $e2c$ such a type of edges. Indeed, the belonging or not of such edges to a spanning tree T prevents the creation of cycles in T. The proposed approach is a separation-construction type in a structured search tree form. The separation is done on the $e2c$-edges of G and induces the construction step to describe constraints (2) of the program (P) which ensures the elimination of cycles relatively to $e2c$-edges. Each branch k of the search tree is constructed by acting alternately on both separation and construction steps, which terminates at a leaf f when there are no $e2c$-edges. Hence, at each of such leaves f, we obtain a subgraph H with two possibilities. In the first, the edges of H constitute a spanning tree reduced to the unique solution of the corresponding branch. The second possibility is that H contains only edge-disjointed cycles in which case the constraints eliminating all these cycles are added to the system of constraints previously established. The obtained set of constraints constitutes a multi-objective linear program (P_f) in binary variables with respect to the objective functions of program (P). The program (P_f) can be solved using one of the exact methods proposed in references [3, 9], generating the set MSTf of the efficient spanning trees and the set corresponding $SNDf$ of non-dominated cost-vectors, associated with the leaf f.

4 Preliminary Results

Let $G = (V, E)$ be a connected and undirected graph of order n and size m. The following propositions allow highlight particular edges of the graph G whose the belonging or not to efficient spanning trees is proved.

Proposition 1 *: Let μ a cycle of G and $e \in \mu$ an edge of G. If all costs vectors of other edges of the cycle μ dominate the one of the edge e, then this edge can not belong to any efficient spanning tree.*

Proof. Let T be a spanning tree and $e \in \mu$, μ a cycle of G such that all costs vectors of other edges of the cycle μ dominate the one of the edge e and assume that $e \in T$ and let $f \in \mu$ and $f \notin T$. Then $T' = T \cup f \setminus e$ is a spanning tree that dominates T because the cost vector of f dominates that of $e \in \mu$. ∎

Proposition 2 *: Let e be an edge of G common to a set Δ of cycles of G, and assume that the cost vector of e dominates all other costs vectors of the edges of cycle Δ, then the edge e belongs to all efficient spanning trees of G.*

Proof. We note $E(\Delta)$ the set of all edges in Δ. If there exists an efficient spanning tree T which does not contain the edge e, then $T' = T \cup e \setminus f, \forall f \in E(\Delta)$, is a spanning tree whose cost vector that dominates the one of the spanning tree T because the edge e dominates the edge $f, \forall f \in E(\Delta)$. ∎

Example 1 *Consider the graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{12, 13, 15, 23, 34, 36, 45, 56\}$. Each edge is provided with three costs as shown in the following matrix of costs:*

$$C = \begin{pmatrix} 0 & 2 & 1 & 2 & 4 & 1 & 3 & 0 \\ 3 & 3 & 2 & 2 & 3 & 1 & 1 & 2 \\ 1 & 0 & 3 & 2 & 4 & 1 & 2 & 0 \end{pmatrix}$$

Reduction
Let the cycle-basis $B = \{\mu_1, \mu_2, \mu_3\}$ with $\mu_1 = \{13, 36, 65, 51\}$, $\mu_2 = \{12, 23, 31\}$ and $\mu_3 = \{43, 36, 65, 54\}$.

- *In the cycle μ_3 the cost vector of the edge (34) is dominated by the other costs vectors of all edges in μ_3. According to **Proposition 1**, this edge is removed from the graph $G = G \setminus (34)$.*
- *Determine then a new cycle-basis $B = B \setminus \mu_3$. and find the set $F = \{e_0 \in E \setminus \exists\mu_l$ and $\mu_{l'} \in B; e_0 \in \mu_l \cap \mu_{l'}\}$.*
  Let the new cycle-basis $B = \{\mu_1, \mu_2\}$ with $F = \{13\}$.

Separation and construction
The separation is done with respect to the edge (13) *creating two nodes. At the NT1-type node of the search tree, we deduce the following system:*

$$(S_1)\begin{cases} x_{13}=1 \\ x_{13}+x_{36}+x_{65}+x_{15}\leq 3 \\ x_{13}+x_{32}+x_{21}\leq 2 \\ x_{12}+x_{13}+x_{15}+x_{23}+x_{36}+x_{45}+x_{65}=5 \\ x_{12},x_{13},x_{15},x_{23},x_{36},x_{45},x_{65}\in\{0,1\} \end{cases}$$

Evaluation
$F=\emptyset$*, then the node 1 is a leaf. By using the method described in [3] for the resolution of multi-objective linear program with binary variables* (P_1) *whose systems of constraints is* (S_1)*, it returns all efficient spanning trees associated with this branch of the search tree:*
$T_1=[12,13,45,65,36], Z(T_1)=(6,10,4)$,
$T_2=[13,23,45,65,36], Z(T_2)=(8,9,5)$,
$T_3=[12,13,15,45,36], Z(T_3)=(4,11,5)$.
Reduction
At the NT2-type node of the search tree, the edge (13) is deleted and the graph $G=G\backslash(13)$*. The coordinates of the ideal point* I *corresponding to this node:* $Z(I)=(4,8,6)$ *which is not dominated by any solutions found previously in the first node, therefore the second node is not fathomed.*
The new base $B=\mu_2$ *was obtained with* $\mu_4=\{12,23,36,65,51\}$.
Separation and construction
The corresponding system of linear constraints is:

$$(S_2)\begin{cases} x_{13}=0 \\ x_{12}+x_{23}+x_{36}+x_{65}+x_{51}\leq 4 \\ x_{12}+x_{13}+x_{15}+x_{23}+x_{36}+x_{45}+x_{65}=5 \\ x_{12},x_{13},x_{15},x_{23},x_{36},x_{45},x_{65}\in\{0,1\} \end{cases}$$

Evaluation
In this leaf too, it uses the method described in [3] for the resolution of multi-objective linear program with binary variables (P_2) *associated with* (S_2) *all efficient spanning trees associated with this branch are:*
$T_4=[12,13,45,65,36], Z(T_4)=(6,9,6)$,
$T_5=[12,15,45,65,36], Z(T_5)=(5,9,7)$,
$T_6=[12,23,15,54,36], Z(T_6)=(4,10,7)$,
$T_7=[15,23,45,56,36], Z(T_7)=(7,8,8)$.
Thus, the final efficient set of spanning trees of G is: $\{T_1,T_2,\cdots,T_7\}$.

5 Conclusion

The main idea of our method is based on a branching process on common edges with at least two cycles of a graph G. This generates a procedure of constructing constraints of non-existence of cycles, which become easy to enumerate after the branching process. At each leaf f of an arbitrary branch k of the search tree, a bivalent linear multi-objective program (P_f) of the minimum spanning tree problem is solved according to the constraints induced along the branch k. The obtained zero-one sparse matrix of constraints is much smaller than the initial set with an exponential number of constraints, leading to a faster resolution. On the other hand, computing the ideal point is easy for our problem and has allowed us not to visit areas that do not contain efficient spanning trees.

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