Les Annales RECITS (http://www.lrecits.usthb.dz)
Vol. 06, 2019, pages 45–52

Diagonal sums in Pascal pyramid $(1, 2, r)$

Hacène Belbachir[1], **Abdelghani Mehdaoui**[2]

[1,2] USTHB, Faculty of Mathematics, RECITS Laboratory
BP 32, El Alia 16111 Bab Ezzouar, Algiers, Algeria

[1]hacenebelbachir@gmail.com hbelbachir@usthb.dz
[2]mehabdelghani@gmail.com amehdaoui@usthb.dz

Abstract: Our purpose is to describe the recurrence relation associated to the sum of diagonal elements laying along a finite ray of certain type crossing the 3 dimensional Pascal pyramid. Further we also give some new examples and the combinatorial interpretation.

Keywords: Trinomial coefficients, Pascal pyramid, linear recurrence

Résumé : Notre objectif est de décrire la récurrence linéaire liée aux sommes des éléments de la direction $(1, 2, r)$ dans la pyramide de Pascal, nous donnons quelques exemples, la récurrence linéaire principale, ainsi que l'interprétation combinatoire.

Mots clés : Coefficients trinomial, pyramide de Pascal, récurrence linéaire.

1 Introduction

It is well-known that the Fibonacci sequence appears as the sum of elements lying over the principal diagonal of Pascal triangle, which is given by

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+k}{k}$$

and it satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$, with the initial value $F_0 = 0$ and $F_1 = 1$. Raab in [9], generalized this result to other diagonals,

$$U_n = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n-rk}{k}$$

and he showed that U_n satisfies, $U_n = U_{n-1} + U_{n-r-1}$. Belbachir et al. in [3], gives the general form and its recurrence for any given direction in Pascal triangle, by

$$V_n = V_n^{(p,q,r)}(x,y) := \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} \binom{n-qk}{p+rk} x^{n-p-(r+q)k} y^{p+rk}$$

and it satisfies,

$$\begin{cases} V_n - x\binom{r}{1}V_{n-1} + x^2\binom{r}{2}V_{n-2} + \cdots + (-1)^r x^r \binom{r}{r} V_{n-r} = y^r\ V_{n-q-r}. \\ V_0 = \cdots = V_p = 0,\ V_k = \binom{k-1}{p} x^{k-p-1} y^p \end{cases}$$

Moreover, the authors and Szalay in [5, 6] introduced the directions in Pascal pyramid and studied the main direction, given by

$$W_n = \sum_{k=0}^{\lfloor n/(r+2) \rfloor} \binom{n-rk}{k, k, n-(r+2)k} t^k z^{n-(r+2)k},$$

and they showed that W_n satisfies a recurrence relation with non constant coefficients. Now we recall the definition of the trinomial coefficients and we give an illustration of Pascal pyramid, we denote them by $\binom{n}{i,j,k}$, where i, j, k are non-negative integers and $i+j+k = n$. It is known that trinomial coefficients appear in the expansion of $(x+y+z)^n$ as

$$(x+y+z)^n = \sum_{i+j+k=n} \binom{n}{i,\ j,\ k} x^i\ y^j\ z^k,$$

moreover, we have

$$\binom{n}{i,j,k} = \frac{n!}{i! \cdot j! \cdot k!}, \quad \text{for } i+j+k = n \text{ and equal 0 otherwise.}$$

They satisfy the following recurrence relation

$$\binom{n}{i,j,k} = \binom{n-1}{i-1,j,k} + \binom{n-1}{i,j-1,k} + \binom{n-1}{i,j,k-1}, \tag{1}$$

Figure 1 shows Pascal pyramid formed by trinomial coefficients, for more details one can see [7, 10, 12, 13].

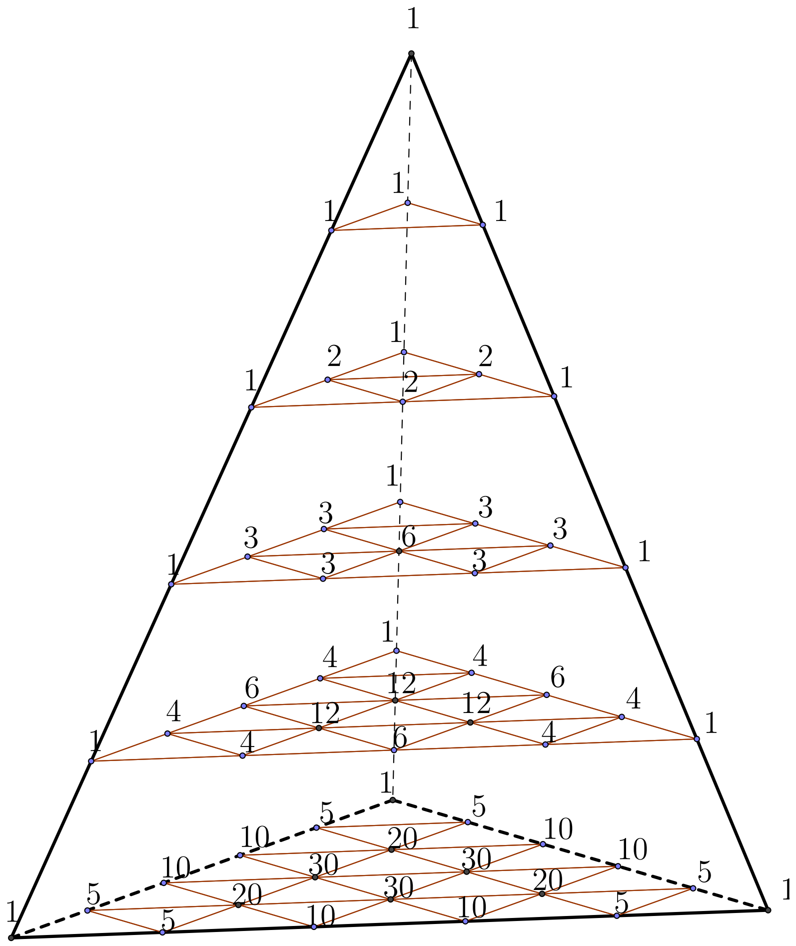

Figure 1: First five levels of Pascal pyramid.

In the sequel, we use the notation in [5].

1.1 Some examples

In this subsection we give some examples

- Putting $r = 0$ and $(x, y, z) = (1, 1, 1)$ we obtain the sequence whith the first values are $(1, 1, 1, 4, 13, 31, 76, ...)$, A116411 in OEIS [11]. Its formula is given by

$$a_n = \sum_{k=0}^{n} \binom{n}{k, 2k, n-2k},$$

and it satisfies

$$\begin{cases} a_0 = a_1 = a_2 = & 1, \\ 2n(2n-3)a_n = & 2(6n^2 - 12n + 5)a_{n-1} - 6(n-1)(2n-3)a_{n-2} \\ & +31(n-2)(n-1)a_{n-3}, \end{cases}$$

and it gives the numbers of paths in a grid from the point $(0, 0)$ to (n, n) using only steps $(0, 1)$, $(1, 1)$ and $(1, 3)$.

- Putting $r = -1, p = 1$ and $(x, y, z) = (1, 1, 1)$ we obtain the sequence whith the first values are $(1, 2, 7, 24, 86, ...)$, A14300 in OEIS [11]. Its formula is given by

$$b_n = \sum_{k=0}^{n} \binom{n}{k, 1+2k, n-2k},$$

and it satisfies

$$\begin{cases} b_0 = 1,\ b_1 = 2,\ b_2 = 7, \\ 2n(3n-4)b_n = (12 - 40n + 21n^2)b_{n-1} + 2(3n-1)(2n-3)b_{n-2}, \end{cases}$$

and it gives the numbers of paths in a grid from the point $(0, 0)$ to (n, n) using only steps $(0, 1)$, $(1, 1)$ and $(0, 2)$.

- Putting $r = -2$ and $(x, y, z) = (1, 1, -1)$ we obtain the sequence whith the first values are $(1, 4, 28, 220, 1816, ...)$, A243116 in OEIS [11]. Its formula is given by

$$c_n = \sum_{k=0}^{n} \binom{n+2k}{k, 2k, n-k} (-1)^{n-k},$$

and it satisfies

$$\begin{cases} c_0 = 1,\ c_1 = 4,\ c_2 = 28, \\ 2n(2n-1)(3n-4)c_n = (3n-2)(39n^2 - 65n + 18)c_{n-1} \\ -2(n-1)(18n^2 - 33n + 10)c_{n-2} + 4(n-2)(n-1)(3n-1)c_{n-3}, \end{cases}$$

2 Main result

In this section we give the main linear recurrence relation.

Theorem 1 *The terms of the sequence $(T_n)_n$ given by*

$$T_n = \sum_{k=0}^{\lfloor n/(r+3) \rfloor} \binom{n-rk}{k, 2k, n-(r+3)k} x^k y^{2k} z^{n-(r+2)k}, \tag{2}$$

satisfy the linear recurrence relation

$$\begin{aligned}
&2n(2n-(r+3))(3n-(r+6))T_n \\
&-z\left(36n^3-(30r+144)n^2+2(3r^2+35r+87)n-4(r^2+8r+15)\right)T_{n-1} \\
&+z^2\left(36n^3-(30r+162)n^2+2(3r^2+43r+117)n-4(2r^2+15r+27)\right)T_{n-2} \\
&-z^3\left(2(n-2)(2n-(r+4))(3n-(r+3))\right)T_{n-3} \\
&=3xy^2(3n-(r+6))(3n-(r+3))(3n-2(r+3))T_{n-r-3} \qquad (3)
\end{aligned}$$

Proof. For $q > -3$, we have

$$\begin{aligned}
&2n(2n-(r+3))(3n-(r+6))T_n \\
&-z\left(36n^3-(30r+144)n^2+2(3r^2+35r+87)n-4(r^2+8r+15)\right)T_{n-1} \\
&+z^2\left(36n^3-(30r+162)n^2+2(3r^2+43r+117)n-4(2r^2+15r+27)\right)T_{n-2} \\
&-z^3\left(2(n-2)(2n-(r+4))(3n-(r+3))\right)T_{n-3} \\
&=2n(2n-(r+3))(3n-(r+6))\sum_k \binom{n-rk}{k,\ 2k,\ n-(r+3)k} x^k\ y^{2k}\ z^{n-(r+3)k} \\
&-\left(36n^3-(30r+144)n^2+2(3r^2+35r+87)n-4(r^2+8r+15)\right) \\
&\sum_k \binom{n-1-rk}{k,\ 2k,\ n-1-(r+3)k} x^k\ y^{2k}\ z^{n-(r+3)k} \\
&+\left(36n^3-(30r+162)n^2+2(3r^2+43r+117)n-4(2r^2+15r+27)\right) \\
&\sum_k \binom{n-2-rk}{k,\ 2k,\ n-2-(r+3)k} x^k\ y^{2k}\ z^{n-(r+4)k} \\
&-\left(2(n-2)(2n-(r+4))(3n-(r+3))\right)\sum_k \binom{n-3-rk}{k,\ 2k,\ n-3-(r+3)k} x^k\ y^{2k}\ z^{n-(r+3)k}
\end{aligned}$$

$$= \sum_k \binom{n-3-rk}{k,\ 2k,\ n-3-(r+3)k} x^k\, y^{2k}\, z^{n-(r+3)k}$$

$$\Bigg[2n(2n-(r+3))(3n-(r+6)) \frac{(n-rk)(n-1-rk)(n-2-rk)}{(n-(r+3)k)(n-1-(r+3)k)(n-2-(r+3)k)}$$

$$- \Bigg(36n^3 - (30r+144)n^2 + 2(3r^2+35r+87)n - 4(r^2+8r+15) \Bigg)$$

$$\frac{(n-1-rk)(n-2-rk)}{(n-1-(r+3)k)(n-2-(r+3)k)}$$

$$+ \Bigg(36n^3 - (30r+162)n^2 + 2(3r^2+43r+117)n - 4(2r^2+15r+27) \Bigg) \frac{(n-2-rk)}{(n-2-(r+3)k)}$$

$$- \Bigg(2(n-2)(2n-(r+4))(3n-(r+3)) \Bigg) \Bigg]$$

$$= \sum_k \frac{(n-3-rk)!}{k!\ 2k!\ (n-3-(r+3)k)!} x^k\, y^{2k}\, z^{n-(r+3)k}$$

$$\frac{6k^2(2k-1)(3n-(r+6))(3n-(r+3))(3n-2(r+3))}{(n-(r+3)k)(n-1-(r+3)k)(n-2-(r+3)k)}$$

$$= 3(3n-(r+6))(3n-(r+3))(3n-2(r+3))$$

$$\sum_k \frac{(n-3-rk)!}{k!\ 2k!\ (n-(r+3)k)!} 2k^2(2k-1) x^k\, y^{2k}\, z^{n-(r+3)k}$$

Let $j = k-1$

$$= 3\, x\, y^2 (3n-(r+6))(3n-(r+3))(3n-2(r+3))$$

$$\sum_j \frac{(n-(r+3)-rj)!}{j!\ 2j!\ (n-(r+3)-(r+3)j)!} x^j\, y^{2j} z^{n-(r+3)-(r+3)j}$$

$$= 3\, x\, y^2 (3n-(r+6))(3n-(r+3))(3n-2(r+3))$$

$$\sum_j \binom{n-(r+3)-rj}{j,\ 2j,\ (n-(r+3)-(r+3)j)} x^j\, y^{2j}\, z^{n-(r+3)-(r+3)j}$$

$$= 3\, x\, y^2 (3n-(r+6))(3n-(r+3))(3n-2(r+3)) T_{n-r-3}$$

■

3 The Morgan-Voyce phenomenon

We remark that the last subscript of the sequence T appears in the right hand side of (3) is $n-r-3$. If $r=-2$, $r=-1$ or $r=0$, this term can be contracted to T_{n-1}, T_{n-2} or T_{n-3} , respectively. That is one of the terms of (3) is perturbed by the coefficient $3xy^2(3n-(q+6))(3n-(q+3))(3n-2(q+3))$ of T_{n-r-3}. This is the Morgan-Voyce phenomenon. Aït-Amrane et *al* [1] studied the analogous question in case of the Pascal triangle and the authors studies it also in [5] in Pascal pyramid.

3.1 Some particular cases

When $r = -2, -1$ *or* 0 the first, second or the third term is perturbed, respectively.

Corollary 2 *For* $r = 0$ *the recurrence relation (3) simplifies*

$$\begin{aligned}&2n(2n-(q+3))(3n-(q+6))T_n = \\ &z\Big(36n^3-(30q+144)n^2+2(3q^2+35q+87)n-4(q^2+8q+15)\Big)T_{n-1} \\ &-z^2\Big(36n^3-(30q+162)n^2+2(3q^2+43q+117)n-4(2q^2+15q+27)\Big)T_{n-2} \\ &+z^2\Big((3n-(q+3))2(n-2)(2n-(q+4))(3n-(q+3))\Big)T_{n-3}\end{aligned} \tag{4}$$

Proof. It follows from 3 by substituting r by 0. ∎

Corollary 3 *For* $r = -1$ *the recurrence relation (3) simplifies*

$$\begin{aligned}&2n(2n-(q+3))(3n-(q+6))T_n = \\ &z\Big(36n^3-(30q+144)n^2+2(3q^2+35q+87)n-4(q^2+8q+15)\Big)T_{n-1} \\ &-z^2\Big(36n^3-(30q+162)n^2+2(3q^2+43q+117)n-4(2q^2+15q+27)\Big)T_{n-2}+(3n-(q+3)) \\ &\Big(z^3 2(n-2)(2n-(q+4))(3n-(q+3))-3xy^2(3n-(q+6))(3n-2(q+3))\Big)T_{n-3}\end{aligned} \tag{5}$$

Proof. It follows from 3 by substituting r by -1. ∎

Corollary 4 *For* $r = -2$ *the recurrence relation (3) simplifies*

$$\begin{aligned}&2n(2n-(q+3))(3n-(q+6))T_n = \\ &z\Big(36n^3-(30q+144)n^2+2(3q^2+35q+87)n-4(q^2+8q+15)\Big)T_{n-1} \\ &-z^2\Big(36n^3-(30q+162)n^2+2(3q^2+43q+117)n-4(2q^2+15q+27)\Big)T_{n-2}+(3n-(q+3)) \\ &\Big(z^3 2(n-2)(2n-(q+4))(3n-(q+3))-3xy^2(3n-(q+6))(3n-2(q+3))\Big)T_{n-3}\end{aligned} \tag{6}$$

Proof. It follows from 3 by substituting r by -2. ∎

3.2 Combinatorial interpretation

A combinatorial interpretation of the sequence (2) corresponding to the direction $(r, 1, 2)$ in the Pascal pyramid is the following. It follows easily from the sum (2).

Theorem 5 *The sequence T_n in (2) counts the sum of products of weights of lattice paths from the point $(0, 0)$ to the point (n, n)using steps $\{(1, 0), (1 + r, 3 + r), (1, 1)\}$, where the first is weighted with y, the second with x and the last one with z.*

Proof. The proof can be obtained easily from the formula 2. ■

References

[1] Ait-Amrane, L., Belbachir, H., Betina, K. (2016). Periods of morgan-voyce sequences and elliptic curves. *Mathematica Slovaca*, 66(6), 1267-1284.

[2] Basil, M., Pascal's pyramid, *Math. Teacher*, 61(1), 1968, 19-21.

[3] Belbachir, H., Komatsu, T., Szalay, L. Characterization of linear recurrences associated to rays in Pascal's triangle. In T. Komatsu (Ed.), *AIP Conference Proceedings*, 1264(1), 2010, 90-99.

[4] Belbachir, H., Komatsu, T. Szalay, L., Linear recurrences associated to rays in Pascal's triangle and combinatorial identities. *Math. Slovaca*, 64, 2014, 287–300.

[5] H. Belbachir, A. Mehdaoui, and L. Szalay, Diagonal sums in Pascal pyramid, *Journal of Combinatorial Theory. Series A*, 165, 2019, 106–116.

[6] H. Belbachir, A. Mehdaoui, and L. Szalay, Diagonal Sums in the Pascal Pyramid , II: Applications, *Journal of Integer Sequences*, 22(3), 2019.

[7] Bondarenko B. A., Generalized Pascal triangles and pyramids, their fractals graphs and applications, *Fibonacci Association*, 1993

[8] Phillipou A. N., A note on the Fibonacci sequence of order K and the multinomial coefficients, 1983.

[9] Raab, J. A., A generalization of the connection between the Fibonacci sequence and Pascal's triangle, *Fibonacci Quart.* 1(3) 1963, 21-31.

[10] Rosenthal, E. B., A Pascal pyramid for trinomial coefficients, *Math. Teacher*, 54, 1961, 336-338.

[11] Sloane, N. J. A., The On-Line Encyclopedia of Integer Sequences. https://oeis.org/.

[12] Staib, J., Staib, L. The Pascal Pyramid, *Math. Teacher* 71, 1978, 505-510.

[13] Zeitler, H., Trinomial formula, Pascal-and Sierpinski-pyramid, *Int. J. Math. Edu. Sci. Techn.*, 33, 2002, 256-266.